

Multi-Precision Math

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This text has been placed in the public domain. This text corresponds to the v0.39 release of the LibTomMath project.

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Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 1.1 | Multiple Precision Arithmetic | 1 |
| 1.1.1 | What is Multiple Precision Arithmetic? | 1 |
| 1.1.2 | The Need for Multiple Precision Arithmetic | 1 |
| 1.1.3 | Benefits of Multiple Precision Arithmetic | 3 |
| 1.2 | Purpose of This Text | 4 |
| 1.3 | Discussion and Notation | 5 |
| 1.3.1 | Notation | 5 |
| 1.3.2 | Precision Notation | 5 |
| 1.3.3 | Algorithm Inputs and Outputs | 6 |
| 1.3.4 | Mathematical Expressions | 6 |
| 1.3.5 | Work Effort | 7 |
| 1.4 | Exercises | 7 |
| 1.5 | Introduction to LibTomMath | 9 |
| 1.5.1 | What is LibTomMath? | 9 |
| 1.5.2 | Goals of LibTomMath | 9 |
| 1.6 | Choice of LibTomMath | 10 |
| 1.6.1 | Code Base | 10 |
| 1.6.2 | API Simplicity | 11 |
| 1.6.3 | Optimizations | 11 |
| 1.6.4 | Portability and Stability | 11 |
| 1.6.5 | Choice | 12 |
| 2 | Getting Started | 13 |
| 2.1 | Library Basics | 13 |
| 2.2 | What is a Multiple Precision Integer? | 14 |
| 2.2.1 | The mp_int Structure | 15 |

| | | |
|----------|--|-----------|
| 2.3 | Argument Passing | 17 |
| 2.4 | Return Values | 18 |
| 2.5 | Initialization and Clearing | 19 |
| 2.5.1 | Initializing an mp_int | 19 |
| 2.5.2 | Clearing an mp_int | 22 |
| 2.6 | Maintenance Algorithms | 24 |
| 2.6.1 | Augmenting an mp_int's Precision | 24 |
| 2.6.2 | Initializing Variable Precision mp_ints | 27 |
| 2.6.3 | Multiple Integer Initializations and Clearings | 29 |
| 2.6.4 | Clamping Excess Digits | 31 |
| 3 | Basic Operations | 35 |
| 3.1 | Introduction | 35 |
| 3.2 | Assigning Values to mp_int Structures | 35 |
| 3.2.1 | Copying an mp_int | 35 |
| 3.2.2 | Creating a Clone | 40 |
| 3.3 | Zeroing an Integer | 41 |
| 3.4 | Sign Manipulation | 42 |
| 3.4.1 | Absolute Value | 42 |
| 3.4.2 | Integer Negation | 43 |
| 3.5 | Small Constants | 45 |
| 3.5.1 | Setting Small Constants | 45 |
| 3.5.2 | Setting Large Constants | 47 |
| 3.6 | Comparisons | 48 |
| 3.6.1 | Unsigned Comparisons | 48 |
| 3.6.2 | Signed Comparisons | 51 |
| 4 | Basic Arithmetic | 55 |
| 4.1 | Introduction | 55 |
| 4.2 | Addition and Subtraction | 56 |
| 4.2.1 | Low Level Addition | 56 |
| 4.2.2 | Low Level Subtraction | 61 |
| 4.2.3 | High Level Addition | 65 |
| 4.2.4 | High Level Subtraction | 68 |
| 4.3 | Bit and Digit Shifting | 71 |
| 4.3.1 | Multiplication by Two | 71 |
| 4.3.2 | Division by Two | 74 |
| 4.4 | Polynomial Basis Operations | 77 |
| 4.4.1 | Multiplication by x | 77 |

| | | |
|----------|---|------------|
| 4.4.2 | Division by x | 80 |
| 4.5 | Powers of Two | 83 |
| 4.5.1 | Multiplication by Power of Two | 83 |
| 4.5.2 | Division by Power of Two | 87 |
| 4.5.3 | Remainder of Division by Power of Two | 90 |
| 5 | Multiplication and Squaring | 95 |
| 5.1 | The Multipliers | 95 |
| 5.2 | Multiplication | 96 |
| 5.2.1 | The Baseline Multiplication | 96 |
| 5.2.2 | Faster Multiplication by the “Comba” Method | 101 |
| 5.2.3 | Polynomial Basis Multiplication | 109 |
| 5.2.4 | Karatsuba Multiplication | 112 |
| 5.2.5 | Toom-Cook 3-Way Multiplication | 118 |
| 5.2.6 | Signed Multiplication | 128 |
| 5.3 | Squaring | 130 |
| 5.3.1 | The Baseline Squaring Algorithm | 131 |
| 5.3.2 | Faster Squaring by the “Comba” Method | 134 |
| 5.3.3 | Polynomial Basis Squaring | 138 |
| 5.3.4 | Karatsuba Squaring | 139 |
| 5.3.5 | Toom-Cook Squaring | 144 |
| 5.3.6 | High Level Squaring | 144 |
| 6 | Modular Reduction | 149 |
| 6.1 | Basics of Modular Reduction | 149 |
| 6.2 | The Barrett Reduction | 150 |
| 6.2.1 | Fixed Point Arithmetic | 150 |
| 6.2.2 | Choosing a Radix Point | 152 |
| 6.2.3 | Trimming the Quotient | 153 |
| 6.2.4 | Trimming the Residue | 154 |
| 6.2.5 | The Barrett Algorithm | 154 |
| 6.2.6 | The Barrett Setup Algorithm | 158 |
| 6.3 | The Montgomery Reduction | 160 |
| 6.3.1 | Digit Based Montgomery Reduction | 163 |
| 6.3.2 | Baseline Montgomery Reduction | 164 |
| 6.3.3 | Faster “Comba” Montgomery Reduction | 169 |
| 6.3.4 | Montgomery Setup | 175 |
| 6.4 | The Diminished Radix Algorithm | 177 |
| 6.4.1 | Choice of Moduli | 179 |

| | | |
|----------|---|------------|
| 6.4.2 | Choice of k | 180 |
| 6.4.3 | Restricted Diminished Radix Reduction | 180 |
| 6.4.4 | Unrestricted Diminished Radix Reduction | 186 |
| 6.5 | Algorithm Comparison | 191 |
| 7 | Exponentiation | 193 |
| 7.1 | Exponentiation Basics | 193 |
| 7.1.1 | Single Digit Exponentiation | 195 |
| 7.2 | k -ary Exponentiation | 198 |
| 7.2.1 | Optimal Values of k | 199 |
| 7.2.2 | Sliding-Window Exponentiation | 199 |
| 7.3 | Modular Exponentiation | 201 |
| 7.3.1 | Barrett Modular Exponentiation | 205 |
| 7.4 | Quick Power of Two | 216 |
| 8 | Higher Level Algorithms | 219 |
| 8.1 | Integer Division with Remainder | 219 |
| 8.1.1 | Quotient Estimation | 221 |
| 8.1.2 | Normalized Integers | 222 |
| 8.1.3 | Radix- β Division with Remainder | 222 |
| 8.2 | Single Digit Helpers | 233 |
| 8.2.1 | Single Digit Addition and Subtraction | 233 |
| 8.2.2 | Single Digit Multiplication | 236 |
| 8.2.3 | Single Digit Division | 239 |
| 8.2.4 | Single Digit Root Extraction | 243 |
| 8.3 | Random Number Generation | 245 |
| 8.4 | Formatted Representations | 248 |
| 8.4.1 | Reading Radix- n Input | 248 |
| 8.4.2 | Generating Radix- n Output | 252 |
| 9 | Number Theoretic Algorithms | 257 |
| 9.1 | Greatest Common Divisor | 257 |
| 9.1.1 | Complete Greatest Common Divisor | 260 |
| 9.2 | Least Common Multiple | 265 |
| 9.3 | Jacobi Symbol Computation | 267 |
| 9.3.1 | Jacobi Symbol | 268 |
| 9.4 | Modular Inverse | 274 |
| 9.4.1 | General Case | 275 |
| 9.5 | Primality Tests | 278 |

| | | |
|-------|---------------------------------|-----|
| 9.5.1 | Trial Division | 278 |
| 9.5.2 | The Fermat Test | 282 |
| 9.5.3 | The Miller-Rabin Test | 284 |

List of Figures

| | | |
|------|---|----|
| 1.1 | Typical Data Types for the C Programming Language | 2 |
| 1.2 | Exercise Scoring System | 8 |
| | | |
| 2.1 | Design Flow of the First Few Original LibTomMath Functions. . | 14 |
| 2.2 | The mp_int Structure | 16 |
| 2.3 | LibTomMath Error Codes | 18 |
| 2.4 | Algorithm mp_init | 19 |
| 2.5 | Algorithm mp_clear | 22 |
| 2.6 | Algorithm mp_grow | 25 |
| 2.7 | Algorithm mp_init_size | 27 |
| 2.8 | Algorithm mp_init_multi | 30 |
| 2.9 | Algorithm mp_clamp | 32 |
| | | |
| 3.1 | Algorithm mp_copy | 36 |
| 3.2 | Algorithm mp_init_copy | 40 |
| 3.3 | Algorithm mp_zero | 41 |
| 3.4 | Algorithm mp_abs | 42 |
| 3.5 | Algorithm mp_neg | 44 |
| 3.6 | Algorithm mp_set | 46 |
| 3.7 | Algorithm mp_set_int | 47 |
| 3.8 | Comparison Return Codes | 49 |
| 3.9 | Algorithm mp_cmp_mag | 49 |
| 3.10 | Algorithm mp_cmp | 51 |
| | | |
| 4.1 | Algorithm s_mp_add | 57 |
| 4.2 | Algorithm s_mp_sub | 62 |
| 4.3 | Algorithm mp_add | 66 |
| 4.4 | Addition Guide Chart | 67 |

| | | |
|------|--|-----|
| 4.5 | Algorithm mp_sub | 69 |
| 4.6 | Subtraction Guide Chart | 69 |
| 4.7 | Algorithm mp_mul_2 | 72 |
| 4.8 | Algorithm mp_div_2 | 75 |
| 4.9 | Algorithm mp_lshd | 78 |
| 4.10 | Sliding Window Movement | 79 |
| 4.11 | Algorithm mp_rshd | 81 |
| 4.12 | Algorithm mp_mul_2d | 84 |
| 4.13 | Algorithm mp_div_2d | 88 |
| 4.14 | Algorithm mp_mod_2d | 91 |
| | | |
| 5.1 | Algorithm s_mp_mul_digs | 97 |
| 5.2 | Long-Hand Multiplication Diagram | 98 |
| 5.3 | Comba Multiplication Diagram | 103 |
| 5.4 | Algorithm Comba Fixup | 103 |
| 5.5 | Algorithm fast_s_mp_mul_digs | 105 |
| 5.6 | Asymptotic Running Time of Polynomial Basis Multiplication . . | 110 |
| 5.7 | Algorithm mp_karatsuba_mul | 113 |
| 5.8 | Algorithm mp_toom_mul | 119 |
| 5.9 | Algorithm mp_toom_mul (continued) | 120 |
| 5.10 | Algorithm mp_mul | 128 |
| 5.11 | Squaring Optimization Diagram | 130 |
| 5.12 | Algorithm s_mp_sqr | 132 |
| 5.13 | Algorithm fast_s_mp_sqr | 135 |
| 5.14 | Algorithm mp_karatsuba_sqr | 140 |
| 5.15 | Algorithm mp_sqr | 145 |
| | | |
| 6.1 | Algorithm mp_reduce | 155 |
| 6.2 | Algorithm mp_reduce_setup | 159 |
| 6.3 | Algorithm Montgomery Reduction | 161 |
| 6.4 | Example of Montgomery Reduction (I) | 161 |
| 6.5 | Algorithm Montgomery Reduction (modified I) | 162 |
| 6.6 | Example of Montgomery Reduction (II) | 162 |
| 6.7 | Algorithm Montgomery Reduction (modified II) | 163 |
| 6.8 | Example of Montgomery Reduction | 164 |
| 6.9 | Algorithm mp_montgomery_reduce | 165 |
| 6.10 | Algorithm fast_mp_montgomery_reduce | 170 |
| 6.11 | Algorithm mp_montgomery_setup | 176 |
| 6.12 | Algorithm Diminished Radix Reduction | 178 |

| | | |
|------|---|-----|
| 6.13 | Example Diminished Radix Reduction | 179 |
| 6.14 | Algorithm mp_dr_reduce | 181 |
| 6.15 | Algorithm mp_dr_setup | 184 |
| 6.16 | Algorithm mp_dr_is_modulus | 185 |
| 6.17 | Algorithm mp_reduce_2k | 187 |
| 6.18 | Algorithm mp_reduce_2k_setup | 188 |
| 6.19 | Algorithm mp_reduce_is_2k | 190 |
| | | |
| 7.1 | Left to Right Exponentiation | 194 |
| 7.2 | Example of Left to Right Exponentiation | 195 |
| 7.3 | Algorithm mp_expt_d | 196 |
| 7.4 | k -ary Exponentiation | 199 |
| 7.5 | Optimal Values of k for k -ary Exponentiation | 200 |
| 7.6 | Optimal Values of k for Sliding Window Exponentiation | 200 |
| 7.7 | Sliding Window k -ary Exponentiation | 201 |
| 7.8 | Algorithm mp_exptmod | 202 |
| 7.9 | Algorithm s_mp_exptmod | 206 |
| 7.10 | Algorithm s_mp_exptmod (continued) | 207 |
| 7.11 | Sliding Window State Diagram | 209 |
| 7.12 | Algorithm mp_2expt | 216 |
| | | |
| 8.1 | Algorithm Radix- β Integer Division | 220 |
| 8.2 | Algorithm mp_div | 223 |
| 8.3 | Algorithm mp_div (continued) | 224 |
| 8.4 | Algorithm mp_add_d | 234 |
| 8.5 | Algorithm mp_mul_d | 237 |
| 8.6 | Algorithm mp_div_d | 240 |
| 8.7 | Algorithm mp_n_root | 244 |
| 8.8 | Algorithm mp_rand | 246 |
| 8.9 | Lower ASCII Map | 249 |
| 8.10 | Algorithm mp_read_radix | 250 |
| 8.11 | Algorithm mp_toradix | 253 |
| 8.12 | Example of Algorithm mp_toradix. | 254 |
| | | |
| 9.1 | Algorithm Greatest Common Divisor (I) | 258 |
| 9.2 | Algorithm Greatest Common Divisor (II) | 258 |
| 9.3 | Algorithm Greatest Common Divisor (III) | 259 |
| 9.4 | Algorithm mp_gcd | 261 |
| 9.5 | Algorithm mp_lcm | 265 |

| | | |
|------|---|-----|
| 9.6 | Algorithm <code>mp_jacobi</code> | 270 |
| 9.7 | Probability of successful trial division to detect non-primes . . . | 280 |
| 9.8 | Algorithm <code>mp_prime_is_divisible</code> | 281 |
| 9.9 | Algorithm <code>mp_prime_fermat</code> | 283 |
| 9.10 | Algorithm <code>mp_prime_miller_rabin</code> | 285 |

Prefaces

When I tell people about my LibTom projects and that I release them as public domain they are often puzzled. They ask why I did it and especially why I continue to work on them for free. The best I can explain it is “Because I can.” Which seems odd and perhaps too terse for adult conversation. I often qualify it with “I am able, I am willing.” which perhaps explains it better. I am the first to admit there is not anything that special with what I have done. Perhaps others can see that too and then we would have a society to be proud of. My LibTom projects are what I am doing to give back to society in the form of tools and knowledge that can help others in their endeavours.

I started writing this book because it was the most logical task to further my goal of open academia. The LibTomMath source code itself was written to be easy to follow and learn from. There are times, however, where pure C source code does not explain the algorithms properly. Hence this book. The book literally starts with the foundation of the library and works itself outwards to the more complicated algorithms. The use of both pseudo-code and verbatim source code provides a duality of “theory” and “practice” that the computer science students of the world shall appreciate. I never deviate too far from relatively straightforward algebra and I hope that this book can be a valuable learning asset.

This book and indeed much of the LibTom projects would not exist in their current form if it was not for a plethora of kind people donating their time, resources and kind words to help support my work. Writing a text of significant length (along with the source code) is a tiresome and lengthy process. Currently the LibTom project is four years old, comprises of literally thousands of users and over 100,000 lines of source code, TeX and other material. People like Mads and Greg were there at the beginning to encourage me to work well. It is amazing how timely validation from others can boost morale to continue the project. Definitely my parents were there for me by providing room and board

during the many months of work in 2003.

To my many friends whom I have met through the years I thank you for the good times and the words of encouragement. I hope I honour your kind gestures with this project.

Open Source. Open Academia. Open Minds.

Tom St Denis

I found the opportunity to work with Tom appealing for several reasons, not only could I broaden my own horizons, but also contribute to educate others facing the problem of having to handle big number mathematical calculations.

This book is Tom's child and he has been caring and fostering the project ever since the beginning with a clear mind of how he wanted the project to turn out. I have helped by proofreading the text and we have had several discussions about the layout and language used.

I hold a masters degree in cryptography from the University of Southern Denmark and have always been interested in the practical aspects of cryptography.

Having worked in the security consultancy business for several years in São Paulo, Brazil, I have been in touch with a great deal of work in which multiple precision mathematics was needed. Understanding the possibilities for speeding up multiple precision calculations is often very important since we deal with outdated machine architecture where modular reductions, for example, become painfully slow.

This text is for people who stop and wonder when first examining algorithms such as RSA for the first time and asks themselves, "You tell me this is only secure for large numbers, fine; but how do you implement these numbers?"

Mads Rasmussen
São Paulo - SP
Brazil

It's all because I broke my leg. That just happened to be at about the same time that Tom asked for someone to review the section of the book about Karatsuba multiplication. I was laid up, alone and immobile, and thought "Why not?" I vaguely knew what Karatsuba multiplication was, but not really, so I thought I could help, learn, and stop myself from watching daytime cable TV, all at once.

At the time of writing this, I've still not met Tom or Mads in meatspace. I've been following Tom's progress since his first splash on the sci.crypt Usenet news group. I watched him go from a clueless newbie, to the cryptographic equivalent of a reformed smoker, to a real contributor to the field, over a period of about two years. I've been impressed with his obvious intelligence, and astounded by his productivity. Of course, he's young enough to be my own child, so he doesn't have my problems with staying awake.

When I reviewed that single section of the book, in its very earliest form, I was very pleasantly surprised. So I decided to collaborate more fully, and at least review all of it, and perhaps write some bits too. There's still a long way to go with it, and I have watched a number of close friends go through the mill of publication, so I think that the way to go is longer than Tom thinks it is. Nevertheless, it's a good effort, and I'm pleased to be involved with it.

Greg Rose, Sydney, Australia, June 2003.

Chapter 1

Introduction

1.1 Multiple Precision Arithmetic

1.1.1 What is Multiple Precision Arithmetic?

When we think of long-hand arithmetic such as addition or multiplication we rarely consider the fact that we instinctively raise or lower the precision of the numbers we are dealing with. For example, in decimal we almost immediately can reason that 7 times 6 is 42. However, 42 has two digits of precision as opposed to one digit we started with. Further multiplications of say 3 result in a larger precision result 126. In these few examples we have multiple precisions for the numbers we are working with. Despite the various levels of precision a single subset¹ of algorithms can be designed to accommodate them.

By way of comparison a fixed or single precision operation would lose precision on various operations. For example, in the decimal system with fixed precision $6 \cdot 7 = 2$.

Essentially at the heart of computer based multiple precision arithmetic are the same long-hand algorithms taught in schools to manually add, subtract, multiply and divide.

1.1.2 The Need for Multiple Precision Arithmetic

The most prevalent need for multiple precision arithmetic, often referred to as “bignum” math, is within the implementation of public-key cryptography

¹With the occasional optimization.

algorithms. Algorithms such as RSA [11] and Diffie-Hellman [12] require integers of significant magnitude to resist known cryptanalytic attacks. For example, at the time of this writing a typical RSA modulus would be at least greater than 10^{309} . However, modern programming languages such as ISO C [18] and Java [19] only provide intrinsic support for integers which are relatively small and single precision.

| Data Type | Range |
|-----------|--|
| char | $-128 \dots 127$ |
| short | $-32768 \dots 32767$ |
| long | $-2147483648 \dots 2147483647$ |
| long long | $-9223372036854775808 \dots 9223372036854775807$ |

Figure 1.1: Typical Data Types for the C Programming Language

The largest data type guaranteed to be provided by the ISO C programming language² can only represent values up to 10^{19} as shown in figure 1.1. On its own the C language is insufficient to accomodate the magnitude required for the problem at hand. An RSA modulus of magnitude 10^{19} could be trivially factored³ on the average desktop computer, rendering any protocol based on the algorithm insecure. Multiple precision algorithms solve this very problem by extending the range of representable integers while using single precision data types.

Most advancements in fast multiple precision arithmetic stem from the need for faster and more efficient cryptographic primitives. Faster modular reduction and exponentiation algorithms such as Barrett's algorithm, which have appeared in various cryptographic journals, can render algorithms such as RSA and Diffie-Hellman more efficient. In fact, several major companies such as RSA Security, Certicom and Entrust have built entire product lines on the implementation and deployment of efficient algorithms.

However, cryptography is not the only field of study that can benefit from fast multiple precision integer routines. Another auxiliary use of multiple precision integers is high precision floating point data types. The basic IEEE [13] standard floating point type is made up of an integer mantissa q , an exponent e and a sign bit s . Numbers are given in the form $n = q \cdot b^e \cdot -1^s$ where $b = 2$

²As per the ISO C standard. However, each compiler vendor is allowed to augment the precision as they see fit.

³A Pollard-Rho factoring would take only 2^{16} time.

is the most common base for IEEE. Since IEEE floating point is meant to be implemented in hardware the precision of the mantissa is often fairly small (23, 48 and 64 bits). The mantissa is merely an integer and a multiple precision integer could be used to create a mantissa of much larger precision than hardware alone can efficiently support. This approach could be useful where scientific applications must minimize the total output error over long calculations.

Yet another use for large integers is within arithmetic on polynomials of large characteristic (i.e. $GF(p)[x]$ for large p). In fact the library discussed within this text has already been used to form a polynomial basis library⁴.

1.1.3 Benefits of Multiple Precision Arithmetic

The benefit of multiple precision representations over single or fixed precision representations is that no precision is lost while representing the result of an operation which requires excess precision. For example, the product of two n -bit integers requires at least $2n$ bits of precision to be represented faithfully. A multiple precision algorithm would augment the precision of the destination to accomodate the result while a single precision system would truncate excess bits to maintain a fixed level of precision.

It is possible to implement algorithms which require large integers with fixed precision algorithms. For example, elliptic curve cryptography (*ECC*) is often implemented on smartcards by fixing the precision of the integers to the maximum size the system will ever need. Such an approach can lead to vastly simpler algorithms which can accomodate the integers required even if the host platform cannot natively accomodate them⁵. However, as efficient as such an approach may be, the resulting source code is not normally very flexible. It cannot, at runtime, accomodate inputs of higher magnitude than the designer anticipated.

Multiple precision algorithms have the most overhead of any style of arithmetic. For the the most part the overhead can be kept to a minimum with careful planning, but overall, it is not well suited for most memory starved platforms. However, multiple precision algorithms do offer the most flexibility in terms of the magnitude of the inputs. That is, the same algorithms based on multiple precision integers can accomodate any reasonable size input without the designer's explicit forethought. This leads to lower cost of ownership for the code as it only has to be written and tested once.

⁴See <http://poly.libtomcrypt.org> for more details.

⁵For example, the average smartcard processor has an 8 bit accumulator.

1.2 Purpose of This Text

The purpose of this text is to instruct the reader regarding how to implement efficient multiple precision algorithms. That is to not only explain a limited subset of the core theory behind the algorithms but also the various “house keeping” elements that are neglected by authors of other texts on the subject. Several well reknowned texts [1, 2] give considerably detailed explanations of the theoretical aspects of algorithms and often very little information regarding the practical implementation aspects.

In most cases how an algorithm is explained and how it is actually implemented are two very different concepts. For example, the Handbook of Applied Cryptography (*HAC*), algorithm 14.7 on page 594, gives a relatively simple algorithm for performing multiple precision integer addition. However, the description lacks any discussion concerning the fact that the two integer inputs may be of differing magnitudes. As a result the implementation is not as simple as the text would lead people to believe. Similarly the division routine (*algorithm 14.20, pp. 598*) does not discuss how to handle sign or handle the dividend’s decreasing magnitude in the main loop (*step #3*).

Both texts also do not discuss several key optimal algorithms required such as “Comba” and Karatsuba multipliers and fast modular inversion, which we consider practical oversights. These optimal algorithms are vital to achieve any form of useful performance in non-trivial applications.

To solve this problem the focus of this text is on the practical aspects of implementing a multiple precision integer package. As a case study the “LibTomMath”⁶ package is used to demonstrate algorithms with real implementations⁷ that have been field tested and work very well. The LibTomMath library is freely available on the Internet for all uses and this text discusses a very large portion of the inner workings of the library.

The algorithms that are presented will always include at least one “pseudo-code” description followed by the actual C source code that implements the algorithm. The pseudo-code can be used to implement the same algorithm in other programming languages as the reader sees fit.

This text shall also serve as a walkthrough of the creation of multiple precision algorithms from scratch. Showing the reader how the algorithms fit together as well as where to start on various taskings.

⁶Available at <http://math.libtomcrypt.com>

⁷In the ISO C programming language.

1.3 Discussion and Notation

1.3.1 Notation

A multiple precision integer of n -digits shall be denoted as $x = (x_{n-1}, \dots, x_1, x_0)_\beta$ and represent the integer $x \equiv \sum_{i=0}^{n-1} x_i \beta^i$. The elements of the array x are said to be the radix β digits of the integer. For example, $x = (1, 2, 3)_{10}$ would represent the integer $1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 = 123$.

The term “mp_int” shall refer to a composite structure which contains the digits of the integer it represents, as well as auxiliary data required to manipulate the data. These additional members are discussed further in section 2.2.1. For the purposes of this text a “multiple precision integer” and an “mp_int” are assumed to be synonymous. When an algorithm is specified to accept an mp_int variable it is assumed the various auxiliary data members are present as well. An expression of the type *variablename.item* implies that it should evaluate to the member named “item” of the variable. For example, a string of characters may have a member “length” which would evaluate to the number of characters in the string. If the string a equals “hello” then it follows that $a.length = 5$.

For certain discussions more generic algorithms are presented to help the reader understand the final algorithm used to solve a given problem. When an algorithm is described as accepting an integer input it is assumed the input is a plain integer with no additional multiple-precision members. That is, algorithms that use integers as opposed to mp_ints as inputs do not concern themselves with the housekeeping operations required such as memory management. These algorithms will be used to establish the relevant theory which will subsequently be used to describe a multiple precision algorithm to solve the same problem.

1.3.2 Precision Notation

The variable β represents the radix of a single digit of a multiple precision integer and must be of the form q^p for $q, p \in \mathbb{Z}^+$. A single precision variable must be able to represent integers in the range $0 \leq x < q\beta$ while a double precision variable must be able to represent integers in the range $0 \leq x < q\beta^2$. The extra radix- q factor allows additions and subtractions to proceed without truncation of the carry. Since all modern computers are binary, it is assumed that q is two.

Within the source code that will be presented for each algorithm, the data type **mp_digit** will represent a single precision integer type, while, the data type **mp_word** will represent a double precision integer type. In several algorithms

(notably the Comba routines) temporary results will be stored in arrays of double precision mp_words. For the purposes of this text x_j will refer to the j 'th digit of a single precision array and \hat{x}_j will refer to the j 'th digit of a double precision array. Whenever an expression is to be assigned to a double precision variable it is assumed that all single precision variables are promoted to double precision during the evaluation. Expressions that are assigned to a single precision variable are truncated to fit within the precision of a single precision data type.

For example, if $\beta = 10^2$ a single precision data type may represent a value in the range $0 \leq x < 10^3$, while a double precision data type may represent a value in the range $0 \leq x < 10^5$. Let $a = 23$ and $b = 49$ represent two single precision variables. The single precision product shall be written as $c \leftarrow a \cdot b$ while the double precision product shall be written as $\hat{c} \leftarrow a \cdot b$. In this particular case, $\hat{c} = 1127$ and $c = 127$. The most significant digit of the product would not fit in a single precision data type and as a result $c \neq \hat{c}$.

1.3.3 Algorithm Inputs and Outputs

Within the algorithm descriptions all variables are assumed to be scalars of either single or double precision as indicated. The only exception to this rule is when variables have been indicated to be of type mp_int. This distinction is important as scalars are often used as array indices and various other counters.

1.3.4 Mathematical Expressions

The $\lfloor \cdot \rfloor$ brackets imply an expression truncated to an integer not greater than the expression itself. For example, $\lfloor 5.7 \rfloor = 5$. Similarly the $\lceil \cdot \rceil$ brackets imply an expression rounded to an integer not less than the expression itself. For example, $\lceil 5.1 \rceil = 6$. Typically when the $/$ division symbol is used the intention is to perform an integer division with truncation. For example, $5/2 = 2$ which will often be written as $\lfloor 5/2 \rfloor = 2$ for clarity. When an expression is written as a fraction a real value division is implied, for example $\frac{5}{2} = 2.5$.

The norm of a multiple precision integer, for example $\|x\|$, will be used to represent the number of digits in the representation of the integer. For example, $\|123\| = 3$ and $\|79452\| = 5$.

1.3.5 Work Effort

To measure the efficiency of the specified algorithms, a modified big-Oh notation is used. In this system all single precision operations are considered to have the same cost⁸. That is a single precision addition, multiplication and division are assumed to take the same time to complete. While this is generally not true in practice, it will simplify the discussions considerably.

Some algorithms have slight advantages over others which is why some constants will not be removed in the notation. For example, a normal baseline multiplication (section 5.2.1) requires $O(n^2)$ work while a baseline squaring (section 5.3) requires $O(\frac{n^2+n}{2})$ work. In standard big-Oh notation these would both be said to be equivalent to $O(n^2)$. However, in the context of this text this is not the case as the magnitude of the inputs will typically be rather small. As a result small constant factors in the work effort will make an observable difference in algorithm efficiency.

All of the algorithms presented in this text have a polynomial time work level. That is, of the form $O(n^k)$ for $n, k \in \mathbb{Z}^+$. This will help make useful comparisons in terms of the speed of the algorithms and how various optimizations will help pay off in the long run.

1.4 Exercises

Within the more advanced chapters a section will be set aside to give the reader some challenging exercises related to the discussion at hand. These exercises are not designed to be prize winning problems, but instead to be thought provoking. Wherever possible the problems are forward minded, stating problems that will be answered in subsequent chapters. The reader is encouraged to finish the exercises as they appear to get a better understanding of the subject material.

That being said, the problems are designed to affirm knowledge of a particular subject matter. Students in particular are encouraged to verify they can answer the problems correctly before moving on.

Similar to the exercises of [1, pp. ix] these exercises are given a scoring system based on the difficulty of the problem. However, unlike [1] the problems do not get nearly as hard. The scoring of these exercises ranges from one (the easiest) to five (the hardest). The following table summarizes the scoring system used.

⁸Except where explicitly noted.

| | |
|-----|--|
| [1] | An easy problem that should only take the reader a manner of minutes to solve. Usually does not involve much computer time to solve. |
| [2] | An easy problem that involves a marginal amount of computer time usage. Usually requires a program to be written to solve the problem. |
| [3] | A moderately hard problem that requires a non-trivial amount of work. Usually involves trivial research and development of new theory from the perspective of a student. |
| [4] | A moderately hard problem that involves a non-trivial amount of work and research, the solution to which will demonstrate a higher mastery of the subject matter. |
| [5] | A hard problem that involves concepts that are difficult for a novice to solve. Solutions to these problems will demonstrate a complete mastery of the given subject. |

Figure 1.2: Exercise Scoring System

Problems at the first level are meant to be simple questions that the reader can answer quickly without programming a solution or devising new theory. These problems are quick tests to see if the material is understood. Problems at the second level are also designed to be easy but will require a program or algorithm to be implemented to arrive at the answer. These two levels are essentially entry level questions.

Problems at the third level are meant to be a bit more difficult than the first two levels. The answer is often fairly obvious but arriving at an exacting solution requires some thought and skill. These problems will almost always involve devising a new algorithm or implementing a variation of another algorithm previously presented. Readers who can answer these questions will feel comfortable with the concepts behind the topic at hand.

Problems at the fourth level are meant to be similar to those of the level three questions except they will require additional research to be completed. The reader will most likely not know the answer right away, nor will the text provide the exact details of the answer until a subsequent chapter.

Problems at the fifth level are meant to be the hardest problems relative to all the other problems in the chapter. People who can correctly answer fifth level problems have a mastery of the subject matter at hand.

Often problems will be tied together. The purpose of this is to start a chain

of thought that will be discussed in future chapters. The reader is encouraged to answer the follow-up problems and try to draw the relevance of problems.

1.5 Introduction to LibTomMath

1.5.1 What is LibTomMath?

LibTomMath is a free and open source multiple precision integer library written entirely in portable ISO C. By portable it is meant that the library does not contain any code that is computer platform dependent or otherwise problematic to use on any given platform.

The library has been successfully tested under numerous operating systems including Unix⁹, MacOS, Windows, Linux, PalmOS and on standalone hardware such as the Gameboy Advance. The library is designed to contain enough functionality to be able to develop applications such as public key cryptosystems and still maintain a relatively small footprint.

1.5.2 Goals of LibTomMath

Libraries which obtain the most efficiency are rarely written in a high level programming language such as C. However, even though this library is written entirely in ISO C, considerable care has been taken to optimize the algorithm implementations within the library. Specifically the code has been written to work well with the GNU C Compiler (*GCC*) on both x86 and ARM processors. Wherever possible, highly efficient algorithms, such as Karatsuba multiplication, sliding window exponentiation and Montgomery reduction have been provided to make the library more efficient.

Even with the nearly optimal and specialized algorithms that have been included the Application Programming Interface (*API*) has been kept as simple as possible. Often generic place holder routines will make use of specialized algorithms automatically without the developer's specific attention. One such example is the generic multiplication algorithm `mp_mul()` which will automatically use Toom-Cook, Karatsuba, Comba or baseline multiplication based on the magnitude of the inputs and the configuration of the library.

Making LibTomMath as efficient as possible is not the only goal of the LibTomMath project. Ideally the library should be source compatible with another popular library which makes it more attractive for developers to use.

⁹All of these trademarks belong to their respective rightful owners.

In this case the MPI library was used as a API template for all the basic functions. MPI was chosen because it is another library that fits in the same niche as LibTomMath. Even though LibTomMath uses MPI as the template for the function names and argument passing conventions, it has been written from scratch by Tom St Denis.

The project is also meant to act as a learning tool for students, the logic being that no easy-to-follow “bignum” library exists which can be used to teach computer science students how to perform fast and reliable multiple precision integer arithmetic. To this end the source code has been given quite a few comments and algorithm discussion points.

1.6 Choice of LibTomMath

LibTomMath was chosen as the case study of this text not only because the author of both projects is one and the same but for more worthy reasons. Other libraries such as GMP [14], MPI [15], LIP [17] and OpenSSL [16] have multiple precision integer arithmetic routines but would not be ideal for this text for reasons that will be explained in the following sub-sections.

1.6.1 Code Base

The LibTomMath code base is all portable ISO C source code. This means that there are no platform dependent conditional segments of code littered throughout the source. This clean and uncluttered approach to the library means that a developer can more readily discern the true intent of a given section of source code without trying to keep track of what conditional code will be used.

The code base of LibTomMath is well organized. Each function is in its own separate source code file which allows the reader to find a given function very quickly. On average there are 76 lines of code per source file which makes the source very easily to follow. By comparison MPI and LIP are single file projects making code tracing very hard. GMP has many conditional code segments which also hinder tracing.

When compiled with GCC for the x86 processor and optimized for speed the entire library is approximately 100KiB¹⁰ which is fairly small compared to GMP (over 250KiB). LibTomMath is slightly larger than MPI (which compiles to about 50KiB) but LibTomMath is also much faster and more complete than MPI.

¹⁰The notation “KiB” means 2^{10} octets, similarly “MiB” means 2^{20} octets.

1.6.2 API Simplicity

LibTomMath is designed after the MPI library and shares the API design. Quite often programs that use MPI will build with LibTomMath without change. The function names correlate directly to the action they perform. Almost all of the functions share the same parameter passing convention. The learning curve is fairly shallow with the API provided which is an extremely valuable benefit for the student and developer alike.

The LIP library is an example of a library with an API that is awkward to work with. LIP uses function names that are often “compressed” to illegible short hand. LibTomMath does not share this characteristic.

The GMP library also does not return error codes. Instead it uses a POSIX.1 [23] signal system where errors are signaled to the host application. This happens to be the fastest approach but definitely not the most versatile. In effect a math error (i.e. invalid input, heap error, etc) can cause a program to stop functioning which is definitely undesirable in many situations.

1.6.3 Optimizations

While LibTomMath is certainly not the fastest library (GMP often beats LibTomMath by a factor of two) it does feature a set of optimal algorithms for tasks such as modular reduction, exponentiation, multiplication and squaring. GMP and LIP also feature such optimizations while MPI only uses baseline algorithms with no optimizations. GMP lacks a few of the additional modular reduction optimizations that LibTomMath features¹¹.

LibTomMath is almost always an order of magnitude faster than the MPI library at computationally expensive tasks such as modular exponentiation. In the grand scheme of “bignum” libraries LibTomMath is faster than the average library and usually slower than the best libraries such as GMP and OpenSSL by only a small factor.

1.6.4 Portability and Stability

LibTomMath will build “out of the box” on any platform equipped with a modern version of the GNU C Compiler (*GCC*). This means that without changes the library will build without configuration or setting up any variables. LIP and MPI will build “out of the box” as well but have numerous known bugs. Most

¹¹At the time of this writing GMP only had Barrett and Montgomery modular reduction algorithms.

notably the author of MPI has recently stopped working on his library and LIP has long since been discontinued.

GMP requires a configuration script to run and will not build out of the box. GMP and LibTomMath are still in active development and are very stable across a variety of platforms.

1.6.5 Choice

LibTomMath is a relatively compact, well documented, highly optimized and portable library which seems only natural for the case study of this text. Various source files from the LibTomMath project will be included within the text. However, the reader is encouraged to download their own copy of the library to actually be able to work with the library.

Chapter 2

Getting Started

2.1 Library Basics

The trick to writing any useful library of source code is to build a solid foundation and work outwards from it. First, a problem along with allowable solution parameters should be identified and analyzed. In this particular case the inability to accomodate multiple precision integers is the problem. Futhermore, the solution must be written as portable source code that is reasonably efficient across several different computer platforms.

After a foundation is formed the remainder of the library can be designed and implemented in a hierarchical fashion. That is, to implement the lowest level dependencies first and work towards the most abstract functions last. For example, before implementing a modular exponentiation algorithm one would implement a modular reduction algorithm. By building outwards from a base foundation instead of using a parallel design methodology the resulting project is highly modular. Being highly modular is a desirable property of any project as it often means the resulting product has a small footprint and updates are easy to perform.

Usually when I start a project I will begin with the header files. I define the data types I think I will need and prototype the initial functions that are not dependent on other functions (within the library). After I implement these base functions I prototype more dependent functions and implement them. The process repeats until I implement all of the functions I require. For example, in the case of LibTomMath I implemented functions such as `mp_init()` well before

I implemented `mp_mul()` and even further before I implemented `mp_exptmod()`. As an example as to why this design works note that the Karatsuba and Toom-Cook multipliers were written *after* the dependent function `mp_exptmod()` was written. Adding the new multiplication algorithms did not require changes to the `mp_exptmod()` function itself and lowered the total cost of ownership (*so to speak*) and of development for new algorithms. This methodology allows new algorithms to be tested in a complete framework with relative ease.



Figure 2.1: Design Flow of the First Few Original LibTomMath Functions.

Only after the majority of the functions were in place did I pursue a less hierarchical approach to auditing and optimizing the source code. For example, one day I may audit the multipliers and the next day the polynomial basis functions.

It only makes sense to begin the text with the preliminary data types and support algorithms required as well. This chapter discusses the core algorithms of the library which are the dependents for every other algorithm.

2.2 What is a Multiple Precision Integer?

Recall that most programming languages, in particular ISO C [18], only have fixed precision data types that on their own cannot be used to represent values

larger than their precision will allow. The purpose of multiple precision algorithms is to use fixed precision data types to create and manipulate multiple precision integers which may represent values that are very large.

As a well known analogy, school children are taught how to form numbers larger than nine by prepending more radix ten digits. In the decimal system the largest single digit value is 9. However, by concatenating digits together larger numbers may be represented. Newly prepended digits (*to the left*) are said to be in a different power of ten column. That is, the number 123 can be described as having a 1 in the hundreds column, 2 in the tens column and 3 in the ones column. Or more formally $123 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0$. Computer based multiple precision arithmetic is essentially the same concept. Larger integers are represented by adjoining fixed precision computer words with the exception that a different radix is used.

What most people probably do not think about explicitly are the various other attributes that describe a multiple precision integer. For example, the integer 154_{10} has two immediately obvious properties. First, the integer is positive, that is the sign of this particular integer is positive as opposed to negative. Second, the integer has three digits in its representation. There is an additional property that the integer possesses that does not concern pencil-and-paper arithmetic. The third property is how many digits placeholders are available to hold the integer.

The human analogy of this third property is ensuring there is enough space on the paper to write the integer. For example, if one starts writing a large number too far to the right on a piece of paper they will have to erase it and move left. Similarly, computer algorithms must maintain strict control over memory usage to ensure that the digits of an integer will not exceed the allowed boundaries. These three properties make up what is known as a multiple precision integer or `mp_int` for short.

2.2.1 The `mp_int` Structure

The `mp_int` structure is the ISO C based manifestation of what represents a multiple precision integer. The ISO C standard does not provide for any such data type but it does provide for making composite data types known as structures. The following is the structure definition used within LibTomMath.

The `mp_int` structure (fig. 2.2) can be broken down as follows.

1. The **used** parameter denotes how many digits of the array **dp** contain the digits used to represent a given integer. The **used** count must be positive

```
typedef struct {
    int used, alloc, sign;
    mp_digit *dp;
} mp_int;
```

Figure 2.2: The mp_int Structure

- (or zero) and may not exceed the **alloc** count.
2. The **alloc** parameter denotes how many digits are available in the array to use by functions before it has to increase in size. When the **used** count of a result would exceed the **alloc** count all of the algorithms will automatically increase the size of the array to accommodate the precision of the result.
 3. The pointer **dp** points to a dynamically allocated array of digits that represent the given multiple precision integer. It is padded with (**alloc** – **used**) zero digits. The array is maintained in a least significant digit order. As a pencil and paper analogy the array is organized such that the right most digits are stored first starting at the location indexed by zero¹ in the array. For example, if **dp** contains $\{a, b, c, \dots\}$ where $\mathbf{dp}_0 = a$, $\mathbf{dp}_1 = b$, $\mathbf{dp}_2 = c$, \dots then it would represent the integer $a + b\beta + c\beta^2 + \dots$
 4. The **sign** parameter denotes the sign as either zero/positive (**MP_ZPOS**) or negative (**MP_NEG**).

Valid mp_int Structures

Several rules are placed on the state of an mp_int structure and are assumed to be followed for reasons of efficiency. The only exceptions are when the structure is passed to initialization functions such as mp_init() and mp_init_copy().

1. The value of **alloc** may not be less than one. That is **dp** always points to a previously allocated array of digits.
2. The value of **used** may not exceed **alloc** and must be greater than or equal to zero.

¹In C all arrays begin at zero.

3. The value of **used** implies the digit at index (*used* − 1) of the **dp** array is non-zero. That is, leading zero digits in the most significant positions must be trimmed.
 - (a) Digits in the **dp** array at and above the **used** location must be zero.
4. The value of **sign** must be **MP_ZPOS** if **used** is zero; this represents the `mp_int` value of zero.

2.3 Argument Passing

A convention of argument passing must be adopted early on in the development of any library. Making the function prototypes consistent will help eliminate many headaches in the future as the library grows to significant complexity. In LibTomMath the multiple precision integer functions accept parameters from left to right as pointers to `mp_int` structures. That means that the source (input) operands are placed on the left and the destination (output) on the right. Consider the following examples.

```
mp_mul(&a, &b, &c);    /* c = a * b */
mp_add(&a, &b, &a);    /* a = a + b */
mp_sqr(&a, &b);        /* b = a * a */
```

The left to right order is a fairly natural way to implement the functions since it lets the developer read aloud the functions and make sense of them. For example, the first function would read “multiply a and b and store in c”.

Certain libraries (*LIP by Lenstra for instance*) accept parameters the other way around, to mimic the order of assignment expressions. That is, the destination (output) is on the left and arguments (inputs) are on the right. In truth, it is entirely a matter of preference. In the case of LibTomMath the convention from the MPI library has been adopted.

Another very useful design consideration, provided for in LibTomMath, is whether to allow argument sources to also be a destination. For example, the second example (*mp_add*) adds *a* to *b* and stores in *a*. This is an important feature to implement since it allows the calling functions to cut down on the number of variables it must maintain. However, to implement this feature specific care has to be given to ensure the destination is not modified before the source is fully read.

2.4 Return Values

A well implemented application, no matter what its purpose, should trap as many runtime errors as possible and return them to the caller. By catching runtime errors a library can be guaranteed to prevent undefined behaviour. However, the end developer can still manage to cause a library to crash. For example, by passing an invalid pointer an application may fault by dereferencing memory not owned by the application.

In the case of LibTomMath the only errors that are checked for are related to inappropriate inputs (division by zero for instance) and memory allocation errors. It will not check that the `mp_int` passed to any function is valid nor will it check pointers for validity. Any function that can cause a runtime error will return an error code as an `int` data type with one of the following values (fig 2.3).

| Value | Meaning |
|----------------|---------------------------------------|
| MP_OKAY | The function was successful |
| MP_VAL | One of the input value(s) was invalid |
| MP_MEM | The function ran out of heap memory |

Figure 2.3: LibTomMath Error Codes

When an error is detected within a function it should free any memory it allocated, often during the initialization of temporary `mp_ints`, and return as soon as possible. The goal is to leave the system in the same state it was when the function was called. Error checking with this style of API is fairly simple.

```
int err;
if ((err = mp_add(&a, &b, &c)) != MP_OKAY) {
    printf("Error: %s\n", mp_error_to_string(err));
    exit(EXIT_FAILURE);
}
```

The GMP [14] library uses C style *signals* to flag errors which is of questionable use. Not all errors are fatal and it was not deemed ideal by the author of LibTomMath to force developers to have signal handlers for such cases.

2.5 Initialization and Clearing

The logical starting point when actually writing multiple precision integer functions is the initialization and clearing of the `mp_int` structures. These two algorithms will be used by the majority of the higher level algorithms.

Given the basic `mp_int` structure an initialization routine must first allocate memory to hold the digits of the integer. Often it is optimal to allocate a sufficiently large pre-set number of digits even though the initial integer will represent zero. If only a single digit were allocated quite a few subsequent re-allocations would occur when operations are performed on the integers. There is a tradeoff between how many default digits to allocate and how many re-allocations are tolerable. Obviously allocating an excessive amount of digits initially will waste memory and become unmanageable.

If the memory for the digits has been successfully allocated then the rest of the members of the structure must be initialized. Since the initial state of an `mp_int` is to represent the zero integer, the allocated digits must be set to zero. The **used** count set to zero and **sign** set to **MP_ZPOS**.

2.5.1 Initializing an `mp_int`

An `mp_int` is said to be initialized if it is set to a valid, preferably default, state such that all of the members of the structure are set to valid values. The `mp_init` algorithm will perform such an action.

Algorithm **mp_init**.

Input. An `mp_int` a

Output. Allocate memory and initialize a to a known valid `mp_int` state.

1. Allocate memory for **MP_PREC** digits.
 2. If the allocation failed return(*MP_MEM*)
 3. for n from 0 to $MP_PREC - 1$ do
 - 3.1 $a_n \leftarrow 0$
 4. $a.sign \leftarrow MP_ZPOS$
 5. $a.used \leftarrow 0$
 6. $a.alloc \leftarrow MP_PREC$
 7. Return(*MP_OKAY*)
-

Figure 2.4: Algorithm `mp_init`

Algorithm mp_init. The purpose of this function is to initialize an mp_int structure so that the rest of the library can properly manipulate it. It is assumed that the input may not have had any of its members previously initialized which is certainly a valid assumption if the input resides on the stack.

Before any of the members such as **sign**, **used** or **alloc** are initialized the memory for the digits is allocated. If this fails the function returns before setting any of the other members. The **MP_PREC** name represents a constant² used to dictate the minimum precision of newly initialized mp_int integers. Ideally, it is at least equal to the smallest precision number you'll be working with.

Allocating a block of digits at first instead of a single digit has the benefit of lowering the number of usually slow heap operations later functions will have to perform in the future. If **MP_PREC** is set correctly the slack memory and the number of heap operations will be trivial.

Once the allocation has been made the digits have to be set to zero as well as the **used**, **sign** and **alloc** members initialized. This ensures that the mp_int will always represent the default state of zero regardless of the original condition of the input.

Remark. This function introduces the idiosyncrasy that all iterative loops, commonly initiated with the “for” keyword, iterate incrementally when the “to” keyword is placed between two expressions. For example, “for a from b to c do” means that a subsequent expression (or body of expressions) are to be evaluated upto $c - b$ times so long as $b \leq c$. In each iteration the variable a is substituted for a new integer that lies inclusively between b and c . If $b > c$ occurred the loop would not iterate. By contrast if the “downto” keyword were used in place of “to” the loop would iterate decrementally.

File: bn_mp_init.c

```

016
017  /* init a new mp_int */
018  int mp_init (mp_int * a)
019  {
020      int i;
021
022      /* allocate memory required and clear it */
023      a->dp = OPT_CAST(mp_digit) XMALLOC (sizeof (mp_digit) * MP_PREC);
024      if (a->dp == NULL) {
025          return MP_MEM;
026      }
027

```

²Defined in the “tommath.h” header file within LibTomMath.

```

028     /* set the digits to zero */
029     for (i = 0; i < MP_PREC; i++) {
030         a->dp[i] = 0;
031     }
032
033     /* set the used to zero, allocated digits to the default precision
034        * and sign to positive */
035     a->used = 0;
036     a->alloc = MP_PREC;
037     a->sign = MP_ZPOS;
038
039     return MP_OKAY;
040 }
041 #endif
042
043 /* ref:          tag: v1.0.1 */
044 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
045 /* commit time: 2017-08-29 22:27:36 +0200 */

```

One immediate observation of this initialization function is that it does not return a pointer to a `mp_int` structure. It is assumed that the caller has already allocated memory for the `mp_int` structure, typically on the application stack. The call to `mp_init()` is used only to initialize the members of the structure to a known default state.

Here we see (line 23) the memory allocation is performed first. This allows us to exit cleanly and quickly if there is an error. If the allocation fails the routine will return **MP_MEM** to the caller to indicate there was a memory error. The function `XMALLOC` is what actually allocates the memory. Technically `XMALLOC` is not a function but a macro defined in “`tommath.h`“. By default, `XMALLOC` will evaluate to `malloc()` which is the C library’s built-in memory allocation routine.

In order to assure the `mp_int` is in a known state the digits must be set to zero. On most platforms this could have been accomplished by using `calloc()` instead of `malloc()`. However, to correctly initialize a integer type to a given value in a portable fashion you have to actually assign the value. The for loop (line 29) performs this required operation.

After the memory has been successfully initialized the remainder of the members are initialized (lines 33 through 34) to their respective default states. At this point the algorithm has succeeded and a success code is returned to the calling function. If this function returns **MP_OKAY** it is safe to assume the `mp_int` structure has been properly initialized and is safe to use with other

functions within the library.

2.5.2 Clearing an mp_int

When an mp_int is no longer required by the application, the memory that has been allocated for its digits must be returned to the application's memory pool with the mp_clear algorithm.

Algorithm **mp_clear**.

Input. An mp_int a

Output. The memory for a shall be deallocated.

1. If a has been previously freed then return(MP_OKAY).
 2. for n from 0 to $a.used - 1$ do
 - 2.1 $a_n \leftarrow 0$
 3. Free the memory allocated for the digits of a .
 4. $a.used \leftarrow 0$
 5. $a.alloc \leftarrow 0$
 6. $a.sign \leftarrow MP_ZPOS$
 7. Return(MP_OKAY).
-

Figure 2.5: Algorithm mp_clear

Algorithm mp_clear. This algorithm accomplishes two goals. First, it clears the digits and the other mp_int members. This ensures that if a developer accidentally re-uses a cleared structure it is less likely to cause problems. The second goal is to free the allocated memory.

The logic behind the algorithm is extended by marking cleared mp_int structures so that subsequent calls to this algorithm will not try to free the memory multiple times. Cleared mp_ints are detectable by having a pre-defined invalid digit pointer **dp** setting.

Once an mp_int has been cleared the mp_int structure is no longer in a valid state for any other algorithm with the exception of algorithms mp_init, mp_init_copy, mp_init_size and mp_clear.

File: bn_mp_clear.c

```
016
017  /* clear one (frees) */
018  void
```

```

019  mp_clear (mp_int * a)
020  {
021      int i;
022
023      /* only do anything if a hasn't been freed previously */
024      if (a->dp != NULL) {
025          /* first zero the digits */
026          for (i = 0; i < a->used; i++) {
027              a->dp[i] = 0;
028          }
029
030          /* free ram */
031          XFREE(a->dp);
032
033          /* reset members to make debugging easier */
034          a->dp = NULL;
035          a->alloc = a->used = 0;
036          a->sign = MP_ZPOS;
037      }
038  }
039  #endif
040
041  /* ref:          tag: v1.0.1 */
042  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
043  /* commit time: 2017-08-29 22:27:36 +0200 */

```

The algorithm only operates on the `mp_int` if it hasn't been previously cleared. The `if` statement (line 24) checks to see if the `dp` member is not **NULL**. If the `mp_int` is a valid `mp_int` then `dp` cannot be **NULL** in which case the `if` statement will evaluate to true.

The digits of the `mp_int` are cleared by the `for` loop (line 26) which assigns a zero to every digit. Similar to `mp_init()` the digits are assigned zero instead of using block memory operations (such as `memset()`) since this is more portable.

The digits are deallocated off the heap via the `XFREE` macro. Similar to `XMALLOC` the `XFREE` macro actually evaluates to a standard C library function. In this case the `free()` function. Since `free()` only deallocates the memory the pointer still has to be reset to **NULL** manually (line 34).

Now that the digits have been cleared and deallocated the other members are set to their final values (lines 35 and 36).

2.6 Maintenance Algorithms

The previous sections describes how to initialize and clear an `mp_int` structure. To further support operations that are to be performed on `mp_int` structures (such as addition and multiplication) the dependent algorithms must be able to augment the precision of an `mp_int` and initialize `mp_ints` with differing initial conditions.

These algorithms complete the set of low level algorithms required to work with `mp_int` structures in the higher level algorithms such as addition, multiplication and modular exponentiation.

2.6.1 Augmenting an `mp_int`'s Precision

When storing a value in an `mp_int` structure, a sufficient number of digits must be available to accomodate the entire result of an operation without loss of precision. Quite often the size of the array given by the `alloc` member is large enough to simply increase the `used` digit count. However, when the size of the array is too small it must be re-sized appropriately to accomodate the result. The `mp_grow` algorithm will provide this functionality.

Algorithm **mp_grow**.

Input. An mp_int a and an integer b .

Output. a is expanded to accomodate b digits.

1. if $a.alloc \geq b$ then return(MP_OKAY)
 2. $u \leftarrow b \pmod{MP_PREC}$
 3. $v \leftarrow b + 2 \cdot MP_PREC - u$
 4. Re-allocate the array of digits a to size v
 5. If the allocation failed then return(MP_MEM).
 6. for n from $a.alloc$ to $v - 1$ do
 - 6.1 $a_n \leftarrow 0$
 7. $a.alloc \leftarrow v$
 8. Return(MP_OKAY)
-

Figure 2.6: Algorithm mp_grow

Algorithm mp_grow. It is ideal to prevent re-allocations from being performed if they are not required (step one). This is useful to prevent mp_ints from growing excessively in code that erroneously calls mp_grow.

The requested digit count is padded up to next multiple of **MP_PREC** plus an additional **MP_PREC** (steps two and three). This helps prevent many trivial reallocations that would grow an mp_int by trivially small values.

It is assumed that the reallocation (step four) leaves the lower $a.alloc$ digits of the mp_int intact. This is much akin to how the *realloc* function from the standard C library works. Since the newly allocated digits are assumed to contain undefined values they are initially set to zero.

File: bn_mp_grow.c

```

016
017  /* grow as required */
018  int mp_grow (mp_int * a, int size)
019  {
020      int      i;
021      mp_digit *tmp;
022
023      /* if the alloc size is smaller alloc more ram */
024      if (a->alloc < size) {
025          /* ensure there are always at least MP_PREC digits extra on top */
026          size += (MP_PREC * 2) - (size % MP_PREC);
027

```

```

028      /* reallocate the array a->dp
029      *
030      * We store the return in a temporary variable
031      * in case the operation failed we don't want
032      * to overwrite the dp member of a.
033      */
034      tmp = OPT_CAST(mp_digit) XREALLOC (a->dp, sizeof (mp_digit) * size);
035      if (tmp == NULL) {
036          /* reallocation failed but "a" is still valid [can be freed] */
037          return MP_MEM;
038      }
039
040      /* reallocation succeeded so set a->dp */
041      a->dp = tmp;
042
043      /* zero excess digits */
044      i      = a->alloc;
045      a->alloc = size;
046      for (; i < a->alloc; i++) {
047          a->dp[i] = 0;
048      }
049  }
050  return MP_OKAY;
051  }
052  #endif
053
054  /* ref:          tag: v1.0.1 */
055  /* git commit:  5953f62e42b24af93748b1ee5e1d062e242c2546 */
056  /* commit time: 2017-08-29 22:27:36 +0200 */

```

A quick optimization is to first determine if a memory re-allocation is required at all. The if statement (line 24) checks if the **alloc** member of the **mp_int** is smaller than the requested digit count. If the count is not larger than **alloc** the function skips the re-allocation part thus saving time.

When a re-allocation is performed it is turned into an optimal request to save time in the future. The requested digit count is padded upwards to 2nd multiple of **MP_PREC** larger than **alloc** (line 26). The **XREALLOC** function is used to re-allocate the memory. As per the other functions **XREALLOC** is actually a macro which evaluates to **realloc** by default. The **realloc** function leaves the base of the allocation intact which means the first **alloc** digits of the **mp_int** are the same as before the re-allocation. All that is left is to clear the newly allocated digits and return.

Note that the re-allocation result is actually stored in a temporary pointer *tmp*. This is to allow this function to return an error with a valid pointer. Earlier releases of the library stored the result of XREALLOC into the *mp_int* *a*. That would result in a memory leak if XREALLOC ever failed.

2.6.2 Initializing Variable Precision *mp_int*s

Occasionally the number of digits required will be known in advance of an initialization, based on, for example, the size of input *mp_int*s to a given algorithm. The purpose of algorithm *mp_init_size* is similar to *mp_init* except that it will allocate *at least* a specified number of digits.

Algorithm ***mp_init_size***.
Input. An *mp_int* *a* and the requested number of digits *b*.
Output. *a* is initialized to hold at least *b* digits.

1. $u \leftarrow b \pmod{MP_PREC}$
2. $v \leftarrow b + 2 \cdot MP_PREC - u$
3. Allocate *v* digits.
4. for *n* from 0 to *v* - 1 do
 - 4.1 $a_n \leftarrow 0$
5. $a.sign \leftarrow MP_ZPOS$
6. $a.used \leftarrow 0$
7. $a.alloc \leftarrow v$
8. Return(*MP_OKAY*)

Figure 2.7: Algorithm *mp_init_size*

Algorithm *mp_init_size*. This algorithm will initialize an *mp_int* structure *a* like algorithm *mp_init* with the exception that the number of digits allocated can be controlled by the second input argument *b*. The input size is padded upwards so it is a multiple of **MP_PREC** plus an additional **MP_PREC** digits. This padding is used to prevent trivial allocations from becoming a bottleneck in the rest of the algorithms.

Like algorithm *mp_init*, the *mp_int* structure is initialized to a default state representing the integer zero. This particular algorithm is useful if it is known ahead of time the approximate size of the input. If the approximation is correct no further memory re-allocations are required to work with the *mp_int*.

File: *bn_mp_init_size.c*

```

016
017 /* init an mp_init for a given size */
018 int mp_init_size (mp_int * a, int size)
019 {
020     int x;
021
022     /* pad size so there are always extra digits */
023     size += (MP_PREC * 2) - (size % MP_PREC);
024
025     /* alloc mem */
026     a->dp = OPT_CAST(mp_digit) XMALLOC (sizeof (mp_digit) * size);
027     if (a->dp == NULL) {
028         return MP_MEM;
029     }
030
031     /* set the members */
032     a->used = 0;
033     a->alloc = size;
034     a->sign = MP_ZPOS;
035
036     /* zero the digits */
037     for (x = 0; x < size; x++) {
038         a->dp[x] = 0;
039     }
040
041     return MP_OKAY;
042 }
043 #endif
044
045 /* ref:          tag: v1.0.1 */
046 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
047 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The number of digits b requested is padded (line 23) by first augmenting it to the next multiple of **MP_PREC** and then adding **MP_PREC** to the result. If the memory can be successfully allocated the `mp_int` is placed in a default state representing the integer zero. Otherwise, the error code **MP_MEM** will be returned (line 28).

The digits are allocated with the `malloc()` function (line 26) and set to zero afterwards (line 37). The **used** count is set to zero, the **alloc** count set to the padded digit count and the **sign** flag set to **MP_ZPOS** to achieve a default valid `mp_int` state (lines 32, 33 and 34). If the function returns successfully

then it is correct to assume that the `mp_int` structure is in a valid state for the remainder of the functions to work with.

2.6.3 Multiple Integer Initializations and Clearings

Occasionally a function will require a series of `mp_int` data types to be made available simultaneously. The purpose of algorithm `mp_init_multi` is to initialize a variable length array of `mp_int` structures in a single statement. It is essentially a shortcut to multiple initializations.

Algorithm **mp_init_multi**.

Input. Variable length array V_k of mp_int variables of length k .

Output. The array is initialized such that each mp_int of V_k is ready to use.

1. for n from 0 to $k - 1$ do
 - 1.1. Initialize the mp_int V_n (*mp_init*)
 - 1.2. If initialization failed then do
 - 1.2.1. for j from 0 to n do
 - 1.2.1.1. Free the mp_int V_j (*mp_clear*)
 - 1.2.2. Return(*MP_MEM*)
 2. Return(*MP_OKAY*)
-

Figure 2.8: Algorithm mp_init_multi

Algorithm mp_init_multi. The algorithm will initialize the array of mp_int variables one at a time. If a runtime error has been detected (*step 1.2*) all of the previously initialized variables are cleared. The goal is an “all or nothing” initialization which allows for quick recovery from runtime errors.

File: bn_mp_init_multi.c

```

016  #include <stdarg.h>
017
018  int mp_init_multi(mp_int *mp, ...)
019  {
020      mp_err res = MP_OKAY;      /* Assume ok until proven otherwise */
021      int n = 0;                 /* Number of ok inits */
022      mp_int* cur_arg = mp;
023      va_list args;
024
025      va_start(args, mp);        /* init args to next argument from caller */
026      while (cur_arg != NULL) {
027          if (mp_init(cur_arg) != MP_OKAY) {
028              /* Oops - error! Back-track and mp_clear what we already
029               succeeded in init-ing, then return error.
030              */
031              va_list clean_args;
032
033              /* now start cleaning up */
034              cur_arg = mp;
035              va_start(clean_args, mp);

```

```

036         while (n-- != 0) {
037             mp_clear(cur_arg);
038             cur_arg = va_arg(clean_args, mp_int*);
039         }
040         va_end(clean_args);
041         res = MP_MEM;
042         break;
043     }
044     n++;
045     cur_arg = va_arg(args, mp_int*);
046 }
047 va_end(args);
048 return res;                /* Assumed ok, if error flagged above. */
049 }
050
051 #endif
052
053 /* ref:          tag: v1.0.1 */
054 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
055 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This function initializes a variable length list of `mp_int` structure pointers. However, instead of having the `mp_int` structures in an actual C array they are simply passed as arguments to the function. This function makes use of the “...” argument syntax of the C programming language. The list is terminated with a final **NULL** argument appended on the right.

The function uses the “`stdarg.h`” *va* functions to step portably through the arguments to the function. A count n of successfully initialized `mp_int` structures is maintained (line 44) such that if a failure does occur, the algorithm can backtrack and free the previously initialized structures (lines 27 to 46).

2.6.4 Clamping Excess Digits

When a function anticipates a result will be n digits it is simpler to assume this is true within the body of the function instead of checking during the computation. For example, a multiplication of a i digit number by a j digit produces a result of at most $i + j$ digits. It is entirely possible that the result is $i + j - 1$ though, with no final carry into the last position. However, suppose the destination had to be first expanded (*via* `mp_grow`) to accomodate $i + j - 1$ digits than further expanded to accomodate the final carry. That would be a considerable waste of time since heap operations are relatively slow.

The ideal solution is to always assume the result is $i+j$ and fix up the **used** count after the function terminates. This way a single heap operation (*at most*) is required. However, if the result was not checked there would be an excess high order zero digit.

For example, suppose the product of two integers was $x_n = (0x_{n-1}x_{n-2}\dots x_0)_\beta$. The leading zero digit will not contribute to the precision of the result. In fact, through subsequent operations more leading zero digits would accumulate to the point the size of the integer would be prohibitive. As a result even though the precision is very low the representation is excessively large.

The `mp_clamp` algorithm is designed to solve this very problem. It will trim high-order zeros by decrementing the **used** count until a non-zero most significant digit is found. Also in this system, zero is considered to be a positive number which means that if the **used** count is decremented to zero, the sign must be set to **MP_ZPOS**.

Algorithm **mp_clamp**.

Input. An `mp_int` a

Output. Any excess leading zero digits of a are removed

1. while $a.used > 0$ and $a_{a.used-1} = 0$ do
 - 1.1 $a.used \leftarrow a.used - 1$
 2. if $a.used = 0$ then do
 - 2.1 $a.sign \leftarrow MP_ZPOS$
-

Figure 2.9: Algorithm `mp_clamp`

Algorithm `mp_clamp`. As can be expected this algorithm is very simple. The loop on step one is expected to iterate only once or twice at the most. For example, this will happen in cases where there is not a carry to fill the last position. Step two fixes the sign for when all of the digits are zero to ensure that the `mp_int` is valid at all times.

File: `bn_mp_clamp.c`

```

016
017  /* trim unused digits
018  *
019  * This is used to ensure that leading zero digits are
020  * trimmed and the leading "used" digit will be non-zero
```



```

021  * Typically very fast. Also fixes the sign if there
022  * are no more leading digits
023  */
024  void
025  mp_clamp (mp_int * a)
026  {
027      /* decrease used while the most significant digit is
028       * zero.
029       */
030      while ((a->used > 0) && (a->dp[a->used - 1] == 0)) {
031          --(a->used);
032      }
033
034      /* reset the sign flag if used == 0 */
035      if (a->used == 0) {
036          a->sign = MP_ZPOS;
037      }
038  }
039 #endif
040
041 /* ref:          tag: v1.0.1 */
042 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
043 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Note on line 27 how to test for the **used** count is made on the left of the **&&** operator. In the C programming language the terms to **&&** are evaluated left to right with a boolean short-circuit if any condition fails. This is important since if the **used** is zero the test on the right would fetch below the array. That is obviously undesirable. The parenthesis on line 30 is used to make sure the **used** count is decremented and not the pointer “a”.

Exercises

- [1] Discuss the relevance of the **used** member of the `mp_int` structure.
- [1] Discuss the consequences of not using padding when performing allocations.
- [2] Estimate an ideal value for **MP_PREC** when performing 1024-bit RSA encryption when $\beta = 2^{28}$.
- [1] Discuss the relevance of the algorithm `mp_clamp`. What does it prevent?
- [1] Give an example of when the algorithm `mp_init_copy` might be useful.

Chapter 3

Basic Operations

3.1 Introduction

In the previous chapter a series of low level algorithms were established that dealt with initializing and maintaining `mp_int` structures. This chapter will discuss another set of seemingly non-algebraic algorithms which will form the low level basis of the entire library. While these algorithm are relatively trivial it is important to understand how they work before proceeding since these algorithms will be used almost intrinsically in the following chapters.

The algorithms in this chapter deal primarily with more “programmer” related tasks such as creating copies of `mp_int` structures, assigning small values to `mp_int` structures and comparisons of the values `mp_int` structures represent.

3.2 Assigning Values to `mp_int` Structures

3.2.1 Copying an `mp_int`

Assigning the value that a given `mp_int` structure represents to another `mp_int` structure shall be known as making a copy for the purposes of this text. The copy of the `mp_int` will be a separate entity that represents the same value as the `mp_int` it was copied from. The `mp_copy` algorithm provides this functionality.

Algorithm **mp_copy**.

Input. An mp_int a and b .

Output. Store a copy of a in b .

1. If $b.alloc < a.used$ then grow b to $a.used$ digits. (*mp_grow*)
 2. for n from 0 to $a.used - 1$ do
 - 2.1 $b_n \leftarrow a_n$
 3. for n from $a.used$ to $b.used - 1$ do
 - 3.1 $b_n \leftarrow 0$
 4. $b.used \leftarrow a.used$
 5. $b.sign \leftarrow a.sign$
 6. return(*MP_OKAY*)
-

Figure 3.1: Algorithm mp_copy

Algorithm mp_copy. This algorithm copies the mp_int a such that upon succesful termination of the algorithm the mp_int b will represent the same integer as the mp_int a . The mp_int b shall be a complete and distinct copy of the mp_int a meaing that the mp_int a can be modified and it shall not affect the value of the mp_int b .

If b does not have enough room for the digits of a it must first have its precision augmented via the mp_grow algorithm. The digits of a are copied over the digits of b and any excess digits of b are set to zero (step two and three). The **used** and **sign** members of a are finally copied over the respective members of b .

Remark. This algorithm also introduces a new idiosyncrasy that will be used throughout the rest of the text. The error return codes of other algorithms are not explicitly checked in the pseudo-code presented. For example, in step one of the mp_copy algorithm the return of mp_grow is not explicitly checked to ensure it succeeded. Text space is limited so it is assumed that if a algorithm fails it will clear all temporarily allocated mp_ints and return the error code itself. However, the C code presented will demonstrate all of the error handling logic required to implement the pseudo-code.

File: bn_mp_copy.c

```

016
017  /* copy, b = a */
018  int
019  mp_copy (mp_int * a, mp_int * b)

```

```
020 {
021     int      res, n;
022
023     /* if dst == src do nothing */
024     if (a == b) {
025         return MP_OKAY;
026     }
027
028     /* grow dest */
029     if (b->alloc < a->used) {
030         if ((res = mp_grow (b, a->used)) != MP_OKAY) {
031             return res;
032         }
033     }
034
035     /* zero b and copy the parameters over */
036     {
037         mp_digit *tmpa, *tmpb;
038
039         /* pointer aliases */
040
041         /* source */
042         tmpa = a->dp;
043
044         /* destination */
045         tmpb = b->dp;
046
047         /* copy all the digits */
048         for (n = 0; n < a->used; n++) {
049             *tmpb++ = *tmpa++;
050         }
051
052         /* clear high digits */
053         for (; n < b->used; n++) {
054             *tmpb++ = 0;
055         }
056     }
057
058     /* copy used count and sign */
059     b->used = a->used;
060     b->sign = a->sign;
061     return MP_OKAY;
```

```

062  }
063  #endif
064
065  /* ref:          tag: v1.0.1 */
066  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
067  /* commit time: 2017-08-29 22:27:36 +0200 */

```

Occasionally a dependent algorithm may copy an `mp_int` effectively into itself such as when the input and output `mp_int` structures passed to a function are one and the same. For this case it is optimal to return immediately without copying digits (line 24).

The `mp_int` *b* must have enough digits to accomodate the used digits of the `mp_int` *a*. If *b.alloc* is less than *a.used* the algorithm `mp_grow` is used to augment the precision of *b* (lines 29 to 33). In order to simplify the inner loop that copies the digits from *a* to *b*, two aliases *tmpa* and *tmpb* point directly at the digits of the `mp_ints` *a* and *b* respectively. These aliases (lines 42 and 45) allow the compiler to access the digits without first dereferencing the `mp_int` pointers and then subsequently the pointer to the digits.

After the aliases are established the digits from *a* are copied into *b* (lines 48 to 50) and then the excess digits of *b* are set to zero (lines 53 to 55). Both “for” loops make use of the pointer aliases and in fact the alias for *b* is carried through into the second “for” loop to clear the excess digits. This optimization allows the alias to stay in a machine register fairly easy between the two loops.

Remarks. The use of pointer aliases is an implementation methodology first introduced in this function that will be used considerably in other functions. Technically, a pointer alias is simply a short hand alias used to lower the number of pointer dereferencing operations required to access data. For example, a for loop may resemble

```

for (x = 0; x < 100; x++) {
    a->num[4]->dp[x] = 0;
}

```

This could be re-written using aliases as

```

mp_digit *tmpa;
a = a->num[4]->dp;
for (x = 0; x < 100; x++) {
    *a++ = 0;
}

```

In this case an alias is used to access the array of digits within an `mp_int` structure directly. It may seem that a pointer alias is strictly not required as a compiler may optimize out the redundant pointer operations. However, there are two dominant reasons to use aliases.

The first reason is that most compilers will not effectively optimize pointer arithmetic. For example, some optimizations may work for the Microsoft Visual C++ compiler (MSVC) and not for the GNU C Compiler (GCC). Also some optimizations may work for GCC and not MSVC. As such it is ideal to find a common ground for as many compilers as possible. Pointer aliases optimize the code considerably before the compiler even reads the source code which means the end compiled code stands a better chance of being faster.

The second reason is that pointer aliases often can make an algorithm simpler to read. Consider the first “for” loop of the function `mp_copy()` re-written to not use pointer aliases.

```
/* copy all the digits */
for (n = 0; n < a->used; n++) {
    b->dp[n] = a->dp[n];
}
```

Whether this code is harder to read depends strongly on the individual. However, it is quantifiably slightly more complicated as there are four variables within the statement instead of just two.

Nested Statements

Another commonly used technique in the source routines is that certain sections of code are nested. This is used in particular with the pointer aliases to highlight code phases. For example, a Comba multiplier (discussed in chapter six) will typically have three different phases. First the temporaries are initialized, then the columns calculated and finally the carries are propagated. In this example the middle column production phase will typically be nested as it uses temporary variables and aliases the most.

The nesting also simplifies the source code as variables that are nested are only valid for their scope. As a result the various temporary variables required do not propagate into other sections of code.

3.2.2 Creating a Clone

Another common operation is to make a local temporary copy of an `mp_int` argument. To initialize an `mp_int` and then copy another existing `mp_int` into the newly initialized `mp_int` will be known as creating a clone. This is useful within functions that need to modify an argument but do not wish to actually modify the original copy. The `mp_init_copy` algorithm has been designed to help perform this task.

Algorithm `mp_init_copy`.
Input. An `mp_int` a and b
Output. a is initialized to be a copy of b .

1. Init a . (*mp_init*)
2. Copy b to a . (*mp_copy*)
3. Return the status of the copy operation.

Figure 3.2: Algorithm `mp_init_copy`

Algorithm `mp_init_copy`. This algorithm will initialize an `mp_int` variable and copy another previously initialized `mp_int` variable into it. As such this algorithm will perform two operations in one step.

File: `bn_mp_init_copy.c`

```

016
017  /* creates "a" then copies b into it */
018  int mp_init_copy (mp_int * a, mp_int * b)
019  {
020      int      res;
021
022      if ((res = mp_init_size (a, b->used)) != MP_OKAY) {
023          return res;
024      }
025
026      if((res = mp_copy (b, a)) != MP_OKAY) {
027          mp_clear(a);
028      }
029
030      return res;
031  }
032  #endif

```



```

033
034  /* ref:          tag: v1.0.1 */
035  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
036  /* commit time:  2017-08-29 22:27:36 +0200 */

```

This will initialize **a** and make it a verbatim copy of the contents of **b**. Note that **a** will have its own memory allocated which means that **b** may be cleared after the call and **a** will be left intact.

3.3 Zeroing an Integer

Resetting an `mp_int` to the default state is a common step in many algorithms. The `mp_zero` algorithm will be the algorithm used to perform this task.

Algorithm **mp_zero**.
Input. An `mp_int` a
Output. Zero the contents of a

1. $a.used \leftarrow 0$
2. $a.sign \leftarrow \text{MP_ZPOS}$
3. for n from 0 to $a.alloc - 1$ do
 - 3.1 $a_n \leftarrow 0$

Figure 3.3: Algorithm `mp_zero`

Algorithm `mp_zero`. This algorithm simply resets a `mp_int` to the default state.

File: `bn_mp_zero.c`

```

016
017  /* set to zero */
018  void mp_zero (mp_int * a)
019  {
020      int      n;
021      mp_digit *tmp;
022
023      a->sign = MP_ZPOS;
024      a->used = 0;
025
026      tmp = a->dp;

```

```

027     for (n = 0; n < a->alloc; n++) {
028         *tmp++ = 0;
029     }
030 }
031 #endif
032
033 /* ref:          tag: v1.0.1 */
034 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
035 /* commit time:  2017-08-29 22:27:36 +0200 */

```

After the function is completed, all of the digits are zeroed, the **used** count is zeroed and the **sign** variable is set to **MP_ZPOS**.

3.4 Sign Manipulation

3.4.1 Absolute Value

With the `mp_int` representation of an integer, calculating the absolute value is trivial. The `mp_abs` algorithm will compute the absolute value of an `mp_int`.

Algorithm `mp_abs`.

Input. An `mp_int` a

Output. Computes $b = |a|$

1. Copy a to b . (*mp_copy*)
 2. If the copy failed return(*MP_MEM*).
 3. $b.\text{sign} \leftarrow \text{MP_ZPOS}$
 4. Return(*MP_OKAY*)
-

Figure 3.4: Algorithm `mp_abs`

Algorithm `mp_abs`. This algorithm computes the absolute of an `mp_int` input. First it copies a over b . This is an example of an algorithm where the check in `mp_copy` that determines if the source and destination are equal proves useful. This allows, for instance, the developer to pass the same `mp_int` as the source and destination to this function without addition logic to handle it.

```

File: bn_mp_abs.c
016
017     /* b = |a|

```

```

018  *
019  * Simple function copies the input and fixes the sign to positive
020  */
021  int
022  mp_abs (mp_int * a, mp_int * b)
023  {
024      int      res;
025
026      /* copy a to b */
027      if (a != b) {
028          if ((res = mp_copy (a, b)) != MP_OKAY) {
029              return res;
030          }
031      }
032
033      /* force the sign of b to positive */
034      b->sign = MP_ZPOS;
035
036      return MP_OKAY;
037  }
038  #endif
039
040  /* ref:          tag: v1.0.1 */
041  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
042  /* commit time: 2017-08-29 22:27:36 +0200 */

```

This fairly trivial algorithm first eliminates non-required duplications (line 27) and then sets the **sign** flag to **MP_ZPOS**.

3.4.2 Integer Negation

With the `mp_int` representation of an integer, calculating the negation is also trivial. The `mp_neg` algorithm will compute the negative of an `mp_int` input.

Algorithm `mp_neg`. This algorithm computes the negation of an input. First it copies *a* over *b*. If *a* has no used digits then the algorithm returns immediately. Otherwise it flips the sign flag and stores the result in *b*. Note that if *a* had no digits then it must be positive by definition. Had step three been omitted then the algorithm would return zero as negative.

```

File: bn_mp_neg.c
016
017  /* b = -a */

```

Algorithm **mp_neg**.

Input. An mp_int a

Output. Computes $b = -a$

1. Copy a to b . (*mp_copy*)
 2. If the copy failed return(*MP_MEM*).
 3. If $a.used = 0$ then return(*MP_OKAY*).
 4. If $a.sign = MP_ZPOS$ then do
 - 4.1 $b.sign = MP_NEG$.
 5. else do
 - 5.1 $b.sign = MP_ZPOS$.
 6. Return(*MP_OKAY*)
-

Figure 3.5: Algorithm mp_neg

```

018 int mp_neg (mp_int * a, mp_int * b)
019 {
020     int      res;
021     if (a != b) {
022         if ((res = mp_copy (a, b)) != MP_OKAY) {
023             return res;
024         }
025     }
026
027     if (mp_iszero(b) != MP_YES) {
028         b->sign = (a->sign == MP_ZPOS) ? MP_NEG : MP_ZPOS;
029     } else {
030         b->sign = MP_ZPOS;
031     }
032
033     return MP_OKAY;
034 }
035 #endif
036
037 /* ref:          tag: v1.0.1 */
038 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
039 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Like `mp_abs()` this function avoids non-required duplications (line 21) and then sets the sign. We have to make sure that only non-zero values get a **sign**

of **MP_NEG**. If the `mp_int` is zero than the **sign** is hard-coded to **MP_ZPOS**.

3.5 Small Constants

3.5.1 Setting Small Constants

Often a `mp_int` must be set to a relatively small value such as 1 or 2. For these cases the `mp_set` algorithm is useful.



Figure 3.6: Algorithm mp_set

Algorithm mp_set. This algorithm sets a mp_int to a small single digit value. Step number 1 ensures that the integer is reset to the default state. The single digit is set (*modulo* β) and the **used** count is adjusted accordingly.

File: bn_mp_set.c

```

016
017  /* set to a digit */
018  void mp_set (mp_int * a, mp_digit b)
019  {
020      mp_zero (a);
021      a->dp[0] = b & MP_MASK;
022      a->used = (a->dp[0] != 0) ? 1 : 0;
023  }
024  #endif
025
026  /* ref:          tag: v1.0.1 */
027  /* git commit: 5953f62e42b24af93748b1ee5e1d062e242c2546 */
028  /* commit time: 2017-08-29 22:27:36 +0200 */

```

First we zero (line 20) the mp_int to make sure that the other members are initialized for a small positive constant. `mp_zero()` ensures that the **sign** is positive and the **used** count is zero. Next we set the digit and reduce it modulo β (line 21). After this step we have to check if the resulting digit is zero or not. If it is not then we set the **used** count to one, otherwise to zero.

We can quickly reduce modulo β since it is of the form 2^k and a quick binary AND operation with $2^k - 1$ will perform the same operation.

One important limitation of this function is that it will only set one digit. The size of a digit is not fixed, meaning source that uses this function should take that into account. Only trivially small constants can be set using this function.

3.5.2 Setting Large Constants

To overcome the limitations of the `mp_set` algorithm the `mp_set_int` algorithm is ideal. It accepts a “long” data type as input and will always treat it as a 32-bit integer.

Algorithm **mp_set_int**.
Input. An `mp_int` a and a “long” integer b
Output. Make a equivalent to b

1. Zero a (*mp_zero*)
 2. for n from 0 to 7 do
 - 2.1 $a \leftarrow a \cdot 16$ (*mp_mul2d*)
 - 2.2 $u \leftarrow \lfloor b/2^{4(7-n)} \rfloor \pmod{16}$
 - 2.3 $a_0 \leftarrow a_0 + u$
 - 2.4 $a.\text{used} \leftarrow a.\text{used} + 1$
 3. Clamp excess used digits (*mp_clamp*)
-

Figure 3.7: Algorithm `mp_set_int`

Algorithm `mp_set_int`. The algorithm performs eight iterations of a simple loop where in each iteration four bits from the source are added to the `mp_int`. Step 2.1 will multiply the current result by sixteen making room for four more bits in the less significant positions. In step 2.2 the next four bits from the source are extracted and are added to the `mp_int`. The **used** digit count is incremented to reflect the addition. The **used** digit counter is incremented since if any of the leading digits were zero the `mp_int` would have zero digits used and the newly added four bits would be ignored.

Excess zero digits are trimmed in steps 2.1 and 3 by using higher level algorithms `mp_mul2d` and `mp_clamp`.

File: `bn_mp_set_int.c`

```

016
017  /* set a 32-bit const */
018  int mp_set_int (mp_int * a, unsigned long b)
019  {
020      int      x, res;
021
022      mp_zero (a);
023
```

```

024  /* set four bits at a time */
025  for (x = 0; x < 8; x++) {
026      /* shift the number up four bits */
027      if ((res = mp_mul_2d (a, 4, a)) != MP_OKAY) {
028          return res;
029      }
030
031      /* OR in the top four bits of the source */
032      a->dp[0] |= (b >> 28) & 15;
033
034      /* shift the source up to the next four bits */
035      b <<= 4;
036
037      /* ensure that digits are not clamped off */
038      a->used += 1;
039  }
040  mp_clamp (a);
041  return MP_OKAY;
042  }
043  #endif
044
045  /* ref:          tag: v1.0.1 */
046  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
047  /* commit time:  2017-08-29 22:27:36 +0200 */

```

This function sets four bits of the number at a time to handle all practical **DIGIT_BIT** sizes. The weird addition on line 38 ensures that the newly added in bits are added to the number of digits. While it may not seem obvious as to why the digit counter does not grow exceedingly large it is because of the shift on line 27 as well as the call to `mp_clamp()` on line 40. Both functions will clamp excess leading digits which keeps the number of used digits low.

3.6 Comparisons

3.6.1 Unsigned Comparisons

Comparing a multiple precision integer is performed with the exact same algorithm used to compare two decimal numbers. For example, to compare 1,234 to 1,264 the digits are extracted by their positions. That is we compare $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$ to $1 \cdot 10^3 + 2 \cdot 10^2 + 6 \cdot 10^1 + 4 \cdot 10^0$ by comparing single digits at a time starting with the highest magnitude positions.

If any leading digit of one integer is greater than a digit in the same position of another integer then obviously it must be greater.

The first comparison routine that will be developed is the unsigned magnitude compare which will perform a comparison based on the digits of two mp_int variables alone. It will ignore the sign of the two inputs. Such a function is useful when an absolute comparison is required or if the signs are known to agree in advance.

To facilitate working with the results of the comparison functions three constants are required.

| Constant | Meaning |
|--------------|--------------|
| MP_GT | Greater Than |
| MP_EQ | Equal To |
| MP_LT | Less Than |

Figure 3.8: Comparison Return Codes

Algorithm **mp_cmp_mag**.

Input. Two mp_ints a and b .

Output. Unsigned comparison results (a to the left of b).

1. If $a.used > b.used$ then return(MP_GT)
 2. If $a.used < b.used$ then return(MP_LT)
 3. for n from $a.used - 1$ to 0 do
 - 3.1 if $a_n > b_n$ then return(MP_GT)
 - 3.2 if $a_n < b_n$ then return(MP_LT)
 4. Return(MP_EQ)
-

Figure 3.9: Algorithm mp_cmp_mag

Algorithm mp_cmp_mag. By saying “ a to the left of b ” it is meant that the comparison is with respect to a , that is if a is greater than b it will return **MP_GT** and similar with respect to when $a = b$ and $a < b$. The first two steps compare the number of digits used in both a and b . Obviously if the digit counts differ there would be an imaginary zero digit in the smaller number where the leading digit of the larger number is. If both have the same number

of digits than the actual digits themselves must be compared starting at the leading digit.

By step three both inputs must have the same number of digits so its safe to start from either $a.used - 1$ or $b.used - 1$ and count down to the zero'th digit. If after all of the digits have been compared, no difference is found, the algorithm returns **MP_EQ**.

File: bn_mp_cmp_mag.c

```

016
017  /* compare magnitude of two ints (unsigned) */
018  int mp_cmp_mag (mp_int * a, mp_int * b)
019  {
020      int      n;
021      mp_digit *tmpa, *tmpb;
022
023      /* compare based on # of non-zero digits */
024      if (a->used > b->used) {
025          return MP_GT;
026      }
027
028      if (a->used < b->used) {
029          return MP_LT;
030      }
031
032      /* alias for a */
033      tmpa = a->dp + (a->used - 1);
034
035      /* alias for b */
036      tmpb = b->dp + (a->used - 1);
037
038      /* compare based on digits */
039      for (n = 0; n < a->used; ++n, --tmpa, --tmpb) {
040          if (*tmpa > *tmpb) {
041              return MP_GT;
042          }
043
044          if (*tmpa < *tmpb) {
045              return MP_LT;
046          }
047      }
048      return MP_EQ;
049  }
```

```

050  #endif
051
052  /* ref:          tag: v1.0.1 */
053  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
054  /* commit time:  2017-08-29 22:27:36 +0200 */

```

The two if statements (lines 24 and 28) compare the number of digits in the two inputs. These two are performed before all of the digits are compared since it is a very cheap test to perform and can potentially save considerable time. The implementation given is also not valid without those two statements. *b.alloc* may be smaller than *a.used*, meaning that undefined values will be read from *b* past the end of the array of digits.

3.6.2 Signed Comparisons

Comparing with sign considerations is also fairly critical in several routines (*division for example*). Based on an unsigned magnitude comparison a trivial signed comparison algorithm can be written.

Algorithm **mp_cmp**.

Input. Two mp_ints *a* and *b*

Output. Signed Comparison Results (*a* to the left of *b*)

1. if *a.sign* = *MP_NEG* and *b.sign* = *MP_ZPOS* then return(*MP_LT*)
 2. if *a.sign* = *MP_ZPOS* and *b.sign* = *MP_NEG* then return(*MP_GT*)
 3. if *a.sign* = *MP_NEG* then
 - 3.1 Return the unsigned comparison of *b* and *a* (*mp_cmp_mag*)
 - 4 Otherwise
 - 4.1 Return the unsigned comparison of *a* and *b*
-

Figure 3.10: Algorithm mp_cmp

Algorithm mp_cmp. The first two steps compare the signs of the two inputs. If the signs do not agree then it can return right away with the appropriate comparison code. When the signs are equal the digits of the inputs must be compared to determine the correct result. In step three the unsigned comparison flips the order of the arguments since they are both negative. For instance, if $-a > -b$ then $|a| < |b|$. Step number four will compare the two when they are both positive.

File: bn_mp_cmp.c

```

016
017  /* compare two ints (signed)*/
018  int
019  mp_cmp (mp_int * a, mp_int * b)
020  {
021      /* compare based on sign */
022      if (a->sign != b->sign) {
023          if (a->sign == MP_NEG) {
024              return MP_LT;
025          } else {
026              return MP_GT;
027          }
028      }
029
030      /* compare digits */
031      if (a->sign == MP_NEG) {
032          /* if negative compare opposite direction */
033          return mp_cmp_mag(b, a);
034      } else {
035          return mp_cmp_mag(a, b);
036      }
037  }
038  #endif
039
040  /* ref:          tag: v1.0.1 */
041  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
042  /* commit time:  2017-08-29 22:27:36 +0200 */

```

The two if statements (lines 22 and 23) perform the initial sign comparison. If the signs are not the equal then which ever has the positive sign is larger. The inputs are compared (line 31) based on magnitudes. If the signs were both negative then the unsigned comparison is performed in the opposite direction (line 33). Otherwise, the signs are assumed to be both positive and a forward direction unsigned comparison is performed.

Exercises

- [2] Modify algorithm `mp_set_int` to accept as input a variable length array of bits.
- [3] Give the probability that algorithm `mp_cmp_mag` will have to compare k digits of two random digits (of equal magnitude) before a difference is found.
- [1] Suggest a simple method to speed up the implementation of `mp_cmp_mag` based on the observations made in the previous problem.

Chapter 4

Basic Arithmetic

4.1 Introduction

At this point algorithms for initialization, clearing, zeroing, copying, comparing and setting small constants have been established. The next logical set of algorithms to develop are addition, subtraction and digit shifting algorithms. These algorithms make use of the lower level algorithms and are the crucial building block for the multiplication algorithms. It is very important that these algorithms are highly optimized. On their own they are simple $O(n)$ algorithms but they can be called from higher level algorithms which easily places them at $O(n^2)$ or even $O(n^3)$ work levels.

All of the algorithms within this chapter make use of the logical bit shift operations denoted by $<<$ and $>>$ for left and right logical shifts respectively. A logical shift is analogous to sliding the decimal point of radix-10 representations. For example, the real number 0.9345 is equivalent to 93.45% which is found by sliding the the decimal two places to the right (*multiplying by $\beta^2 = 10^2$*). Algebraically a binary logical shift is equivalent to a division or multiplication by a power of two. For example, $a << k = a \cdot 2^k$ while $a >> k = \lfloor a/2^k \rfloor$.

One significant difference between a logical shift and the way decimals are shifted is that digits below the zero'th position are removed from the number. For example, consider $1101_2 >> 1$ using decimal notation this would produce 110.1_2 . However, with a logical shift the result is 110_2 .

4.2 Addition and Subtraction

In common twos complement fixed precision arithmetic negative numbers are easily represented by subtraction from the modulus. For example, with 32-bit integers $a - b \pmod{2^{32}}$ is the same as $a + (2^{32} - b) \pmod{2^{32}}$ since $2^{32} \equiv 0 \pmod{2^{32}}$. As a result subtraction can be performed with a trivial series of logical operations and an addition.

However, in multiple precision arithmetic negative numbers are not represented in the same way. Instead a sign flag is used to keep track of the sign of the integer. As a result signed addition and subtraction are actually implemented as conditional usage of lower level addition or subtraction algorithms with the sign fixed up appropriately.

The lower level algorithms will add or subtract integers without regard to the sign flag. That is they will add or subtract the magnitude of the integers respectively.

4.2.1 Low Level Addition

An unsigned addition of multiple precision integers is performed with the same long-hand algorithm used to add decimal numbers. That is to add the trailing digits first and propagate the resulting carry upwards. Since this is a lower level algorithm the name will have a “s_” prefix. Historically that convention stems from the MPI library where “s_” stood for static functions that were hidden from the developer entirely.

Algorithm **s_mp_add**.

Input. Two mp_ints a and b

Output. The unsigned addition $c = |a| + |b|$.

1. if $a.used > b.used$ then
 - 1.1 $min \leftarrow b.used$
 - 1.2 $max \leftarrow a.used$
 - 1.3 $x \leftarrow a$
 2. else
 - 2.1 $min \leftarrow a.used$
 - 2.2 $max \leftarrow b.used$
 - 2.3 $x \leftarrow b$
 3. If $c.alloc < max + 1$ then grow c to hold at least $max + 1$ digits (*mp-grow*)
 4. $oldused \leftarrow c.used$
 5. $c.used \leftarrow max + 1$
 6. $u \leftarrow 0$
 7. for n from 0 to $min - 1$ do
 - 7.1 $c_n \leftarrow a_n + b_n + u$
 - 7.2 $u \leftarrow c_n >> lg(\beta)$
 - 7.3 $c_n \leftarrow c_n \pmod{\beta}$
 8. if $min \neq max$ then do
 - 8.1 for n from min to $max - 1$ do
 - 8.1.1 $c_n \leftarrow x_n + u$
 - 8.1.2 $u \leftarrow c_n >> lg(\beta)$
 - 8.1.3 $c_n \leftarrow c_n \pmod{\beta}$
 9. $c_{max} \leftarrow u$
 10. if $olduse > max$ then
 - 10.1 for n from $max + 1$ to $oldused - 1$ do
 - 10.1.1 $c_n \leftarrow 0$
 11. Clamp excess digits in c . (*mp_clamp*)
 12. Return(*MP_OKAY*)
-

Figure 4.1: Algorithm s_mp_add

Algorithm s_mp_add. This algorithm is loosely based on algorithm 14.7 of HAC [2, pp. 594] but has been extended to allow the inputs to have different magnitudes. Coincidentally the description of algorithm A in Knuth [1, pp. 266] shares the same deficiency as the algorithm from [2]. Even the MIX pseudo machine code presented by Knuth [1, pp. 266-267] is incapable of handling inputs which are of different magnitudes.

The first thing that has to be accomplished is to sort out which of the two

inputs is the largest. The addition logic will simply add all of the smallest input to the largest input and store that first part of the result in the destination. Then it will apply a simpler addition loop to excess digits of the larger input.

The first two steps will handle sorting the inputs such that *min* and *max* hold the digit counts of the two inputs. The variable *x* will be an `mp_int` alias for the largest input or the second input *b* if they have the same number of digits. After the inputs are sorted the destination *c* is grown as required to accomodate the sum of the two inputs. The original **used** count of *c* is copied and set to the new used count.

At this point the first addition loop will go through as many digit positions that both inputs have. The carry variable μ is set to zero outside the loop. Inside the loop an “addition” step requires three statements to produce one digit of the summand. First two digits from *a* and *b* are added together along with the carry μ . The carry of this step is extracted and stored in μ and finally the digit of the result c_n is truncated within the range $0 \leq c_n < \beta$.

Now all of the digit positions that both inputs have in common have been exhausted. If *min* \neq *max* then *x* is an alias for one of the inputs that has more digits. A simplified addition loop is then used to essentially copy the remaining digits and the carry to the destination.

The final carry is stored in c_{max} and digits above *max* upto *oldused* are zeroed which completes the addition.

File: `bn_s_mp_add.c`

```

016
017  /* low level addition, based on HAC pp.594, Algorithm 14.7 */
018  int
019  s_mp_add (mp_int * a, mp_int * b, mp_int * c)
020  {
021      mp_int *x;
022      int      olduse, res, min, max;
023
024      /* find sizes, we let |a| <= |b| which means we have to sort
025       * them. "x" will point to the input with the most digits
026       */
027      if (a->used > b->used) {
028          min = b->used;
029          max = a->used;
030          x = a;
031      } else {
032          min = a->used;
033          max = b->used;

```

```

034     x = b;
035 }
036
037 /* init result */
038 if (c->alloc < (max + 1)) {
039     if ((res = mp_grow (c, max + 1)) != MP_OKAY) {
040         return res;
041     }
042 }
043
044 /* get old used digit count and set new one */
045 olduse = c->used;
046 c->used = max + 1;
047
048 {
049     mp_digit u, *tmpa, *tmpb, *tmpc;
050     int i;
051
052     /* alias for digit pointers */
053
054     /* first input */
055     tmpa = a->dp;
056
057     /* second input */
058     tmpb = b->dp;
059
060     /* destination */
061     tmpc = c->dp;
062
063     /* zero the carry */
064     u = 0;
065     for (i = 0; i < min; i++) {
066         /* Compute the sum at one digit, T[i] = A[i] + B[i] + U */
067         *tmpc = *tmpa++ + *tmpb++ + u;
068
069         /* U = carry bit of T[i] */
070         u = *tmpc >> ((mp_digit)DIGIT_BIT);
071
072         /* take away carry bit from T[i] */
073         *tmpc++ &= MP_MASK;
074     }
075

```

```

076      /* now copy higher words if any, that is in A+B
077      * if A or B has more digits add those in
078      */
079      if (min != max) {
080          for (; i < max; i++) {
081              /* T[i] = X[i] + U */
082              *tmpc = x->dp[i] + u;
083
084              /* U = carry bit of T[i] */
085              u = *tmpc >> ((mp_digit)DIGIT_BIT);
086
087              /* take away carry bit from T[i] */
088              *tmpc++ &= MP_MASK;
089          }
090      }
091
092      /* add carry */
093      *tmpc++ = u;
094
095      /* clear digits above oldused */
096      for (i = c->used; i < oldused; i++) {
097          *tmpc++ = 0;
098      }
099  }
100
101      mp_clamp(c);
102      return MP_OKAY;
103  }
104  #endif
105
106      /* ref:          tag: v1.0.1 */
107      /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
108      /* commit time: 2017-08-29 22:27:36 +0200 */

```

We first sort (lines 27 to 35) the inputs based on magnitude and determine the *min* and *max* variables. Note that *x* is a pointer to an *mp_int* assigned to the largest input, in effect it is a local alias. Next we grow the destination (37 to 42) ensure that it can accomodate the result of the addition.

Similar to the implementation of *mp_copy* this function uses the braced code and local aliases coding style. The three aliases that are on lines 55, 58 and 61 represent the two inputs and destination variables respectively. These aliases are used to ensure the compiler does not have to dereference *a*, *b* or *c* (respectively)

to access the digits of the respective `mp_int`.

The initial carry u will be cleared (line 64), note that u is of type `mp_digit` which ensures type compatibility within the implementation. The initial addition (line 65 to 74) adds digits from both inputs until the smallest input runs out of digits. Similarly the conditional addition loop (line 80 to 90) adds the remaining digits from the larger of the two inputs. The addition is finished with the final carry being stored in $tmpc$ (line 93). Note the “++” operator within the same expression. After line 93, $tmpc$ will point to the $c.used$ ’th digit of the `mp_int` c . This is useful for the next loop (line 96 to 99) which set any old upper digits to zero.

4.2.2 Low Level Subtraction

The low level unsigned subtraction algorithm is very similar to the low level unsigned addition algorithm. The principle difference is that the unsigned subtraction algorithm requires the result to be positive. That is when computing $a - b$ the condition $|a| \geq |b|$ must be met for this algorithm to function properly. Keep in mind this low level algorithm is not meant to be used in higher level algorithms directly. This algorithm as will be shown can be used to create functional signed addition and subtraction algorithms.

For this algorithm a new variable is required to make the description simpler. Recall from section 1.3.1 that a `mp_digit` must be able to represent the range $0 \leq x < 2\beta$ for the algorithms to work correctly. However, it is allowable that a `mp_digit` represent a larger range of values. For this algorithm we will assume that the variable γ represents the number of bits available in a `mp_digit` (*this implies $2^\gamma > \beta$*).

For example, the default for LibTomMath is to use a “unsigned long” for the `mp_digit` “type” while $\beta = 2^{28}$. In ISO C an “unsigned long” data type must be able to represent $0 \leq x < 2^{32}$ meaning that in this case $\gamma \geq 32$.

Algorithm **s_mp_sub**.

Input. Two mp_ints a and b ($|a| \geq |b|$)

Output. The unsigned subtraction $c = |a| - |b|$.

1. $min \leftarrow b.used$
 2. $max \leftarrow a.used$
 3. If $c.alloc < max$ then grow c to hold at least max digits. (*mp_grow*)
 4. $oldused \leftarrow c.used$
 5. $c.used \leftarrow max$
 6. $u \leftarrow 0$
 7. for n from 0 to $min - 1$ do
 - 7.1 $c_n \leftarrow a_n - b_n - u$
 - 7.2 $u \leftarrow c_n >> (\gamma - 1)$
 - 7.3 $c_n \leftarrow c_n \pmod{\beta}$
 8. if $min < max$ then do
 - 8.1 for n from min to $max - 1$ do
 - 8.1.1 $c_n \leftarrow a_n - u$
 - 8.1.2 $u \leftarrow c_n >> (\gamma - 1)$
 - 8.1.3 $c_n \leftarrow c_n \pmod{\beta}$
 9. if $oldused > max$ then do
 - 9.1 for n from max to $oldused - 1$ do
 - 9.1.1 $c_n \leftarrow 0$
 10. Clamp excess digits of c . (*mp_clamp*).
 11. Return(*MP_OKAY*).
-

Figure 4.2: Algorithm s_mp_sub

Algorithm s_mp_sub. This algorithm performs the unsigned subtraction of two mp_int variables under the restriction that the result must be positive. That is when passing variables a and b the condition that $|a| \geq |b|$ must be met for the algorithm to function correctly. This algorithm is loosely based on algorithm 14.9 [2, pp. 595] and is similar to algorithm S in [1, pp. 267] as well. As was the case of the algorithm s_mp_add both other references lack discussion concerning various practical details such as when the inputs differ in magnitude.

The initial sorting of the inputs is trivial in this algorithm since a is guaranteed to have at least the same magnitude of b . Steps 1 and 2 set the *min* and *max* variables. Unlike the addition routine there is guaranteed to be no carry which means that the final result can be at most *max* digits in length as opposed to $max + 1$. Similar to the addition algorithm the **used** count of c is copied locally and set to the maximal count for the operation.

The subtraction loop that begins on step seven is essentially the same as the addition loop of algorithm `s_mp_add` except single precision subtraction is used instead. Note the use of the γ variable to extract the carry (*also known as the borrow*) within the subtraction loops. Under the assumption that two's complement single precision arithmetic is used this will successfully extract the desired carry.

For example, consider subtracting 0101_2 from 0100_2 where $\gamma = 4$ and $\beta = 2$. The least significant bit will force a carry upwards to the third bit which will be set to zero after the borrow. After the very first bit has been subtracted $4 - 1 \equiv 0011_2$ will remain, When the third bit of 0101_2 is subtracted from the result it will cause another carry. In this case though the carry will be forced to propagate all the way to the most significant bit.

Recall that $\beta < 2^\gamma$. This means that if a carry does occur just before the $\lg(\beta)$ 'th bit it will propagate all the way to the most significant bit. Thus, the high order bits of the `mp_digit` that are not part of the actual digit will either be all zero, or all one. All that is needed is a single zero or one bit for the carry. Therefore a single logical shift right by $\gamma - 1$ positions is sufficient to extract the carry. This method of carry extraction may seem awkward but the reason for it becomes apparent when the implementation is discussed.

If b has a smaller magnitude than a then step 9 will force the carry and copy operation to propagate through the larger input a into c . Step 10 will ensure that any leading digits of c above the max 'th position are zeroed.

File: `bn_s_mp_sub.c`

```

016
017  /* low level subtraction (assumes |a| > |b|), HAC pp.595 Algorithm 14.9 */
018  int
019  s_mp_sub (mp_int * a, mp_int * b, mp_int * c)
020  {
021      int      olduse, res, min, max;
022
023      /* find sizes */
024      min = b->used;
025      max = a->used;
026
027      /* init result */
028      if (c->alloc < max) {
029          if ((res = mp_grow (c, max)) != MP_OKAY) {
030              return res;
031          }
032      }

```

```

033     olduse = c->used;
034     c->used = max;
035
036     {
037         mp_digit u, *tmpa, *tmpb, *tmpc;
038         int i;
039
040         /* alias for digit pointers */
041         tmpa = a->dp;
042         tmpb = b->dp;
043         tmpc = c->dp;
044
045         /* set carry to zero */
046         u = 0;
047         for (i = 0; i < min; i++) {
048             /* T[i] = A[i] - B[i] - U */
049             *tmpc = (*tmpa++ - *tmpb++) - u;
050
051             /* U = carry bit of T[i]
052              * Note this saves performing an AND operation since
053              * if a carry does occur it will propagate all the way to the
054              * MSB. As a result a single shift is enough to get the carry
055              */
056             u = *tmpc >> ((mp_digit)((CHAR_BIT * sizeof(mp_digit)) - 1));
057
058             /* Clear carry from T[i] */
059             *tmpc++ &= MP_MASK;
060         }
061
062         /* now copy higher words if any, e.g. if A has more digits than B */
063         for (; i < max; i++) {
064             /* T[i] = A[i] - U */
065             *tmpc = *tmpa++ - u;
066
067             /* U = carry bit of T[i] */
068             u = *tmpc >> ((mp_digit)((CHAR_BIT * sizeof(mp_digit)) - 1));
069
070             /* Clear carry from T[i] */
071             *tmpc++ &= MP_MASK;
072         }
073
074         /* clear digits above used (since we may not have grown result above) */

```



```

075     for (i = c->used; i < olduse; i++) {
076         *tmpc++ = 0;
077     }
078 }
079
080 mp_clamp (c);
081 return MP_OKAY;
082 }
083
084 #endif
085
086 /* ref:          tag: v1.0.1 */
087 /* git commit: 5953f62e42b24af93748b1ee5e1d062e242c2546 */
088 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Like low level addition we “sort” the inputs. Except in this case the sorting is hardcoded (lines 24 and 25). In reality the *min* and *max* variables are only aliases and are only used to make the source code easier to read. Again the pointer alias optimization is used within this algorithm. The aliases *tmpa*, *tmpb* and *tmpc* are initialized (lines 41, 42 and 43) for *a*, *b* and *c* respectively.

The first subtraction loop (lines 46 through 60) subtract digits from both inputs until the smaller of the two inputs has been exhausted. As remarked earlier there is an implementation reason for using the “awkward” method of extracting the carry (line 56). The traditional method for extracting the carry would be to shift by $\lg(\beta)$ positions and logically AND the least significant bit. The AND operation is required because all of the bits above the $\lg(\beta)$ ’th bit will be set to one after a carry occurs from subtraction. This carry extraction requires two relatively cheap operations to extract the carry. The other method is to simply shift the most significant bit to the least significant bit thus extracting the carry with a single cheap operation. This optimization only works on twos compliment machines which is a safe assumption to make.

If *a* has a larger magnitude than *b* an additional loop (lines 63 through 72) is required to propagate the carry through *a* and copy the result to *c*.

4.2.3 High Level Addition

Now that both lower level addition and subtraction algorithms have been established an effective high level signed addition algorithm can be established. This high level addition algorithm will be what other algorithms and developers will use to perform addition of *mp_int* data types.

Recall from section 5.2 that an `mp_int` represents an integer with an unsigned mantissa (*the array of digits*) and a **sign** flag. A high level addition is actually performed as a series of eight separate cases which can be optimized down to three unique cases.

Algorithm `mp_add`.
Input. Two `mp_ints` a and b
Output. The signed addition $c = a + b$.

1. if $a.sign = b.sign$ then do
 - 1.1 $c.sign \leftarrow a.sign$
 - 1.2 $c \leftarrow |a| + |b|$ (*s_mp_add*)
2. else do
 - 2.1 if $|a| < |b|$ then do (*mp_cmp_mag*)
 - 2.1.1 $c.sign \leftarrow b.sign$
 - 2.1.2 $c \leftarrow |b| - |a|$ (*s_mp_sub*)
 - 2.2 else do
 - 2.2.1 $c.sign \leftarrow a.sign$
 - 2.2.2 $c \leftarrow |a| - |b|$
3. Return(*MP_OKAY*).

Figure 4.3: Algorithm `mp_add`

Algorithm `mp_add`. This algorithm performs the signed addition of two `mp_int` variables. There is no reference algorithm to draw upon from either [1] or [2] since they both only provide unsigned operations. The algorithm is fairly straightforward but restricted since subtraction can only produce positive results.

Figure 4.4 lists all of the eight possible input combinations and is sorted to show that only three specific cases need to be handled. The return code of the unsigned operations at step 1.2, 2.1.2 and 2.2.2 are forwarded to step three to check for errors. This simplifies the description of the algorithm considerably and best follows how the implementation actually was achieved.

Also note how the **sign** is set before the unsigned addition or subtraction is performed. Recall from the descriptions of algorithms `s_mp_add` and `s_mp_sub` that the `mp_clamp` function is used at the end to trim excess digits. The `mp_clamp` algorithm will set the **sign** to `MP_ZPOS` when the **used** digit count reaches zero.

| Sign of a | Sign of b | $ a > b $ | Unsigned Operation | Result Sign Flag |
|-------------|-------------|-------------|--------------------|------------------|
| + | + | Yes | $c = a + b$ | $a.sign$ |
| + | + | No | $c = a + b$ | $a.sign$ |
| - | - | Yes | $c = a + b$ | $a.sign$ |
| - | - | No | $c = a + b$ | $a.sign$ |
| | | | | |
| + | - | No | $c = b - a$ | $b.sign$ |
| - | + | No | $c = b - a$ | $b.sign$ |
| | | | | |
| + | - | Yes | $c = a - b$ | $a.sign$ |
| - | + | Yes | $c = a - b$ | $a.sign$ |

Figure 4.4: Addition Guide Chart

For example, consider performing $-a + a$ with algorithm `mp.add`. By the description of the algorithm the sign is set to **MP_NEG** which would produce a result of -0 . However, since the sign is set first then the unsigned addition is performed the subsequent usage of algorithm `mp.clamp` within algorithm `s_mp.add` will force -0 to become 0 .

File: `bn_mp.add.c`

```

016
017  /* high level addition (handles signs) */
018  int mp_add (mp_int * a, mp_int * b, mp_int * c)
019  {
020      int      sa, sb, res;
021
022      /* get sign of both inputs */
023      sa = a->sign;
024      sb = b->sign;
025
026      /* handle two cases, not four */
027      if (sa == sb) {
028          /* both positive or both negative */
029          /* add their magnitudes, copy the sign */
030          c->sign = sa;
031          res = s_mp_add (a, b, c);
032      } else {
033          /* one positive, the other negative */
034          /* subtract the one with the greater magnitude from */

```

```

035     /* the one of the lesser magnitude. The result gets */
036     /* the sign of the one with the greater magnitude. */
037     if (mp_cmp_mag (a, b) == MP_LT) {
038         c->sign = sb;
039         res = s_mp_sub (b, a, c);
040     } else {
041         c->sign = sa;
042         res = s_mp_sub (a, b, c);
043     }
044 }
045 return res;
046 }
047
048 #endif
049
050 /* ref:          tag: v1.0.1 */
051 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
052 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The source code follows the algorithm fairly closely. The most notable new source code addition is the usage of the *res* integer variable which is used to pass result of the unsigned operations forward. Unlike in the algorithm, the variable *res* is merely returned as is without explicitly checking it and returning the constant **MP_OKAY**. The observation is this algorithm will succeed or fail only if the lower level functions do so. Returning their return code is sufficient.

4.2.4 High Level Subtraction

The high level signed subtraction algorithm is essentially the same as the high level signed addition algorithm.

Algorithm **mp_sub**.

Input. Two mp_ints a and b

Output. The signed subtraction $c = a - b$.

1. if $a.sign \neq b.sign$ then do
 - 1.1 $c.sign \leftarrow a.sign$
 - 1.2 $c \leftarrow |a| + |b|$ (s_mp_add)
 2. else do
 - 2.1 if $|a| \geq |b|$ then do (mp_cmp_mag)
 - 2.1.1 $c.sign \leftarrow a.sign$
 - 2.1.2 $c \leftarrow |a| - |b|$ (s_mp_sub)
 - 2.2 else do
 - 2.2.1 $c.sign \leftarrow \begin{cases} MP_ZPOS & \text{if } a.sign = MP_NEG \\ MP_NEG & \text{otherwise} \end{cases}$
 - 2.2.2 $c \leftarrow |b| - |a|$
 3. Return(MP_OKAY).
-

Figure 4.5: Algorithm mp_sub

Algorithm mp_sub. This algorithm performs the signed subtraction of two inputs. Similar to algorithm mp_add there is no reference in either [1] or [2]. Also this algorithm is restricted by algorithm s_mp_sub. Chart 4.6 lists the eight possible inputs and the operations required.

| Sign of a | Sign of b | $ a \geq b $ | Unsigned Operation | Result Sign Flag |
|-------------|-------------|----------------|--------------------|----------------------|
| + | − | Yes | $c = a + b$ | $a.sign$ |
| + | − | No | $c = a + b$ | $a.sign$ |
| − | + | Yes | $c = a + b$ | $a.sign$ |
| − | + | No | $c = a + b$ | $a.sign$ |
| | | | | |
| + | + | Yes | $c = a - b$ | $a.sign$ |
| − | − | Yes | $c = a - b$ | $a.sign$ |
| | | | | |
| + | + | No | $c = b - a$ | opposite of $a.sign$ |
| − | − | No | $c = b - a$ | opposite of $a.sign$ |

Figure 4.6: Subtraction Guide Chart

Similar to the case of algorithm `mp_add` the **sign** is set first before the unsigned addition or subtraction. That is to prevent the algorithm from producing $-a - -a = -0$ as a result.

File: `bn_mp_sub.c`

```

016
017  /* high level subtraction (handles signs) */
018  int
019  mp_sub (mp_int * a, mp_int * b, mp_int * c)
020  {
021      int      sa, sb, res;
022
023      sa = a->sign;
024      sb = b->sign;
025
026      if (sa != sb) {
027          /* subtract a negative from a positive, OR */
028          /* subtract a positive from a negative. */
029          /* In either case, ADD their magnitudes, */
030          /* and use the sign of the first number. */
031          c->sign = sa;
032          res = s_mp_add (a, b, c);
033      } else {
034          /* subtract a positive from a positive, OR */
035          /* subtract a negative from a negative. */
036          /* First, take the difference between their */
037          /* magnitudes, then... */
038          if (mp_cmp_mag (a, b) != MP_LT) {
039              /* Copy the sign from the first */
040              c->sign = sa;
041              /* The first has a larger or equal magnitude */
042              res = s_mp_sub (a, b, c);
043          } else {
044              /* The result has the *opposite* sign from */
045              /* the first number. */
046              c->sign = (sa == MP_ZPOS) ? MP_NEG : MP_ZPOS;
047              /* The second has a larger magnitude */
048              res = s_mp_sub (b, a, c);
049          }
050      }
051      return res;
052  }
```

```

053
054  #endif
055
056  /* ref:          tag: v1.0.1 */
057  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
058  /* commit time:  2017-08-29 22:27:36 +0200 */

```

Much like the implementation of algorithm `mp_add` the variable `res` is used to catch the return code of the unsigned addition or subtraction operations and forward it to the end of the function. On line 38 the “not equal to” `MP_LT` expression is used to emulate a “greater than or equal to” comparison.

4.3 Bit and Digit Shifting

It is quite common to think of a multiple precision integer as a polynomial in x , that is $y = f(\beta)$ where $f(x) = \sum_{i=0}^{n-1} a_i x^i$. This notation arises within discussion of Montgomery and Diminished Radix Reduction as well as Karatsuba multiplication and squaring.

In order to facilitate operations on polynomials in x as above a series of simple “digit” algorithms have to be established. That is to shift the digits left or right as well to shift individual bits of the digits left and right. It is important to note that not all “shift” operations are on radix- β digits.

4.3.1 Multiplication by Two

In a binary system where the radix is a power of two multiplication by two not only arises often in other algorithms it is a fairly efficient operation to perform. A single precision logical shift left is sufficient to multiply a single digit by two.

Algorithm **mp_mul_2**.

Input. One mp_int a

Output. $b = 2a$.

1. If $b.alloc < a.used + 1$ then grow b to hold $a.used + 1$ digits. (*mp-grow*)
 2. $oldused \leftarrow b.used$
 3. $b.used \leftarrow a.used$
 4. $r \leftarrow 0$
 5. for n from 0 to $a.used - 1$ do
 - 5.1 $rr \leftarrow a_n \gg (lg(\beta) - 1)$
 - 5.2 $b_n \leftarrow (a_n \ll 1) + r \pmod{\beta}$
 - 5.3 $r \leftarrow rr$
 6. If $r \neq 0$ then do
 - 6.1 $b_{n+1} \leftarrow r$
 - 6.2 $b.used \leftarrow b.used + 1$
 7. If $b.used < oldused - 1$ then do
 - 7.1 for n from $b.used$ to $oldused - 1$ do
 - 7.1.1 $b_n \leftarrow 0$
 8. $b.sign \leftarrow a.sign$
 9. Return(*MP_OKAY*).
-

Figure 4.7: Algorithm mp_mul_2

Algorithm mp_mul_2. This algorithm will quickly multiply a mp_int by two provided β is a power of two. Neither [1] nor [2] describe such an algorithm despite the fact it arises often in other algorithms. The algorithm is setup much like the lower level algorithm s_mp_add since it is for all intents and purposes equivalent to the operation $b = |a| + |a|$.

Step 1 and 2 grow the input as required to accomodate the maximum number of **used** digits in the result. The initial **used** count is set to $a.used$ at step 4. Only if there is a final carry will the **used** count require adjustment.

Step 6 is an optimization implementation of the addition loop for this specific case. That is since the two values being added together are the same there is no need to perform two reads from the digits of a . Step 6.1 performs a single precision shift on the current digit a_n to obtain what will be the carry for the next iteration. Step 6.2 calculates the n 'th digit of the result as single precision shift of a_n plus the previous carry. Recall from section 4.1 that $a_n \ll 1$ is equivalent to $a_n \cdot 2$. An iteration of the addition loop is finished with forwarding the carry to the next iteration.

Step 7 takes care of any final carry by setting the $a.used$ 'th digit of the

result to the carry and augmenting the **used** count of *b*. Step 8 clears any leading digits of *b* in case it originally had a larger magnitude than *a*.

File: bn_mp_mul_2.c

```

016
017  /* b = a*2 */
018  int mp_mul_2(mp_int * a, mp_int * b)
019  {
020      int      x, res, oldused;
021
022      /* grow to accomodate result */
023      if (b->alloc < (a->used + 1)) {
024          if ((res = mp_grow (b, a->used + 1)) != MP_OKAY) {
025              return res;
026          }
027      }
028
029      oldused = b->used;
030      b->used = a->used;
031
032      {
033          mp_digit r, rr, *tmpa, *tmpb;
034
035          /* alias for source */
036          tmpa = a->dp;
037
038          /* alias for dest */
039          tmpb = b->dp;
040
041          /* carry */
042          r = 0;
043          for (x = 0; x < a->used; x++) {
044
045              /* get what will be the *next* carry bit from the
046               * MSB of the current digit
047               */
048              rr = *tmpa >> ((mp_digit)(DIGIT_BIT - 1));
049
050              /* now shift up this digit, add in the carry [from the previous] */
051              *tmpb++ = ((*tmpa++ << ((mp_digit)1)) | r) & MP_MASK;
052
053              /* copy the carry that would be from the source
054               * digit into the next iteration

```

```

055         */
056         r = rr;
057     }
058
059     /* new leading digit? */
060     if (r != 0) {
061         /* add a MSB which is always 1 at this point */
062         *tmpb = 1;
063         ++(b->used);
064     }
065
066     /* now zero any excess digits on the destination
067      * that we didn't write to
068     */
069     tmpb = b->dp + b->used;
070     for (x = b->used; x < oldused; x++) {
071         *tmpb++ = 0;
072     }
073 }
074 b->sign = a->sign;
075 return MP_OKAY;
076 }
077 #endif
078
079 /* ref:          tag: v1.0.1 */
080 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
081 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This implementation is essentially an optimized implementation of `s_mp_add` for the case of doubling an input. The only noteworthy difference is the use of the logical shift operator on line 51 to perform a single precision doubling.

4.3.2 Division by Two

A division by two can just as easily be accomplished with a logical shift right as multiplication by two can be with a logical shift left.

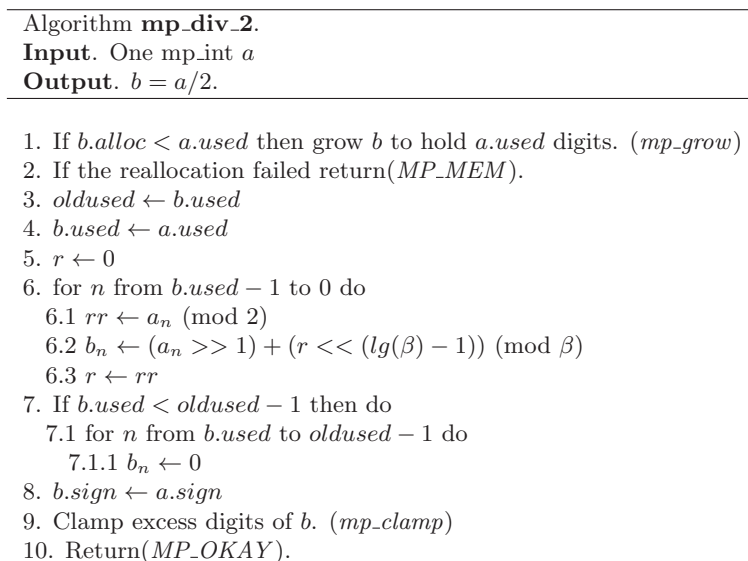


Figure 4.8: Algorithm mp_div_2

Algorithm mp_div_2. This algorithm will divide an mp_int by two using logical shifts to the right. Like mp_mul_2 it uses a modified low level addition core as the basis of the algorithm. Unlike mp_mul_2 the shift operations work from the leading digit to the trailing digit. The algorithm could be written to work from the trailing digit to the leading digit however, it would have to stop one short of $a.used - 1$ digits to prevent reading past the end of the array of digits.

Essentially the loop at step 6 is similar to that of mp_mul_2 except the logical shifts go in the opposite direction and the carry is at the least significant bit not the most significant bit.

File: bn_mp_div_2.c

```

016
017  /* b = a/2 */
018  int mp_div_2(mp_int * a, mp_int * b)
019  {
020      int      x, res, oldused;
021
022      /* copy */

```

```

023     if (b->alloc < a->used) {
024         if ((res = mp_grow (b, a->used)) != MP_OKAY) {
025             return res;
026         }
027     }
028
029     oldused = b->used;
030     b->used = a->used;
031     {
032         mp_digit r, rr, *tmpa, *tmpb;
033
034         /* source alias */
035         tmpa = a->dp + b->used - 1;
036
037         /* dest alias */
038         tmpb = b->dp + b->used - 1;
039
040         /* carry */
041         r = 0;
042         for (x = b->used - 1; x >= 0; x--) {
043             /* get the carry for the next iteration */
044             rr = *tmpa & 1;
045
046             /* shift the current digit, add in carry and store */
047             *tmpb-- = (*tmpa-- >> 1) | (r << (DIGIT_BIT - 1));
048
049             /* forward carry to next iteration */
050             r = rr;
051         }
052
053         /* zero excess digits */
054         tmpb = b->dp + b->used;
055         for (x = b->used; x < oldused; x++) {
056             *tmpb++ = 0;
057         }
058     }
059     b->sign = a->sign;
060     mp_clamp (b);
061     return MP_OKAY;
062 }
063 #endif
064

```

```

065  /* ref:          tag: v1.0.1 */
066  /* git commit:    5953f62e42b24af93748b1ee5e1d062e242c2546 */
067  /* commit time:   2017-08-29 22:27:36 +0200 */

```

4.4 Polynomial Basis Operations

Recall from section 4.3 that any integer can be represented as a polynomial in x as $y = f(\beta)$. Such a representation is also known as the polynomial basis [3, pp. 48]. Given such a notation a multiplication or division by x amounts to shifting whole digits a single place. The need for such operations arises in several other higher level algorithms such as Barrett and Montgomery reduction, integer division and Karatsuba multiplication.

Converting from an array of digits to polynomial basis is very simple. Consider the integer $y \equiv (a_2, a_1, a_0)_\beta$ and recall that $y = \sum_{i=0}^2 a_i \beta^i$. Simply replace β with x and the expression is in polynomial basis. For example, $f(x) = 8x + 9$ is the polynomial basis representation for 89 using radix ten. That is, $f(10) = 8(10) + 9 = 89$.

4.4.1 Multiplication by x

Given a polynomial in x such as $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ multiplying by x amounts to shifting the coefficients up one degree. In this case $f(x) \cdot x = a_n x^{n+1} + a_{n-1} x^n + \dots + a_0 x$. From a scalar basis point of view multiplying by x is equivalent to multiplying by the integer β .

Algorithm **mp_lshd**.

Input. One mp_int a and an integer b

Output. $a \leftarrow a \cdot \beta^b$ (equivalent to multiplication by x^b).

1. If $b \leq 0$ then return(MP_OKAY).
 2. If $a.alloc < a.used + b$ then grow a to at least $a.used + b$ digits. (mp_grow).
 3. If the reallocation failed return(MP_MEM).
 4. $a.used \leftarrow a.used + b$
 5. $i \leftarrow a.used - 1$
 6. $j \leftarrow a.used - 1 - b$
 7. for n from $a.used - 1$ to b do
 - 7.1 $a_i \leftarrow a_j$
 - 7.2 $i \leftarrow i - 1$
 - 7.3 $j \leftarrow j - 1$
 8. for n from 0 to $b - 1$ do
 - 8.1 $a_n \leftarrow 0$
 9. Return(MP_OKAY).
-

Figure 4.9: Algorithm mp_lshd

Algorithm mp_lshd. This algorithm multiplies an mp_int by the b 'th power of x . This is equivalent to multiplying by β^b . The algorithm differs from the other algorithms presented so far as it performs the operation in place instead storing the result in a separate location. The motivation behind this change is due to the way this function is typically used. Algorithms such as mp_add store the result in an optionally different third mp_int because the original inputs are often still required. Algorithm mp_lshd (*and similarly algorithm mp_rshd*) is typically used on values where the original value is no longer required. The algorithm will return success immediately if $b \leq 0$ since the rest of algorithm is only valid when $b > 0$.

First the destination a is grown as required to accomodate the result. The counters i and j are used to form a *sliding window* over the digits of a of length b . The head of the sliding window is at i (*the leading digit*) and the tail at j (*the trailing digit*). The loop on step 7 copies the digit from the tail to the head. In each iteration the window is moved down one digit. The last loop on step 8 sets the lower b digits to zero.

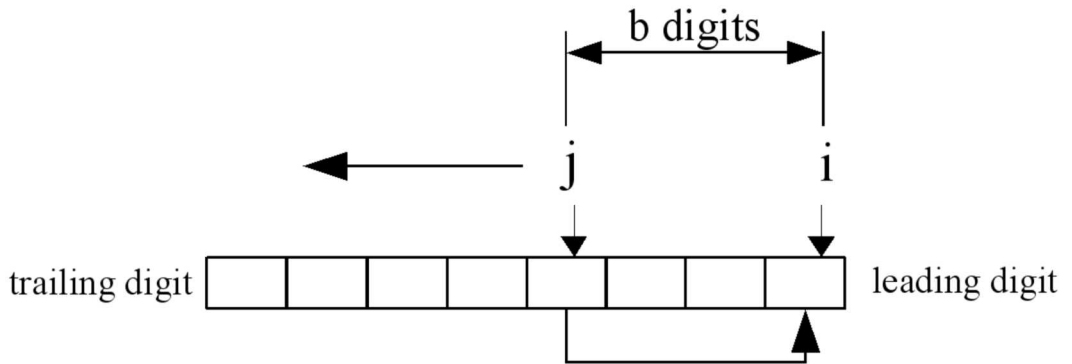


Figure 4.10: Sliding Window Movement

File: bn_mp_lshd.c

```

016
017  /* shift left a certain amount of digits */
018  int mp_lshd (mp_int * a, int b)
019  {
020      int      x, res;
021
022      /* if its less than zero return */
023      if (b <= 0) {
024          return MP_OKAY;
025      }
026
027      /* grow to fit the new digits */
028      if (a->alloc < (a->used + b)) {
029          if ((res = mp_grow (a, a->used + b)) != MP_OKAY) {
030              return res;
031          }
032      }
033
034      {
035          mp_digit *top, *bottom;
036
037          /* increment the used by the shift amount then copy upwards */
038          a->used += b;

```

```

039
040     /* top */
041     top = a->dp + a->used - 1;
042
043     /* base */
044     bottom = (a->dp + a->used - 1) - b;
045
046     /* much like mp_rshd this is implemented using a sliding window
047      * except the window goes the otherway around. Copying from
048      * the bottom to the top. see bn_mp_rshd.c for more info.
049      */
050     for (x = a->used - 1; x >= b; x--) {
051         *top-- = *bottom--;
052     }
053
054     /* zero the lower digits */
055     top = a->dp;
056     for (x = 0; x < b; x++) {
057         *top++ = 0;
058     }
059 }
060 return MP_OKAY;
061 }
062 #endif
063
064 /* ref:          tag: v1.0.1 */
065 /* git commit: 5953f62e42b24af93748b1ee5e1d062e242c2546 */
066 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The if statement (line 23) ensures that the b variable is greater than zero since we do not interpret negative shift counts properly. The **used** count is incremented by b before the copy loop begins. This eliminates the need for an additional variable in the for loop. The variable *top* (line 41) is an alias for the leading digit while *bottom* (line 44) is an alias for the trailing edge. The aliases form a window of exactly b digits over the input.

4.4.2 Division by x

Division by powers of x is easily achieved by shifting the digits right and removing any that will end up to the right of the zero'th digit.

Algorithm **mp_rshd**.

Input. One mp_int a and an integer b

Output. $a \leftarrow a/\beta^b$ (Divide by x^b).

1. If $b \leq 0$ then return.
 2. If $a.\text{used} \leq b$ then do
 - 2.1 Zero a . (mp_zero).
 - 2.2 Return.
 3. $i \leftarrow 0$
 4. $j \leftarrow b$
 5. for n from 0 to $a.\text{used} - b - 1$ do
 - 5.1 $a_i \leftarrow a_j$
 - 5.2 $i \leftarrow i + 1$
 - 5.3 $j \leftarrow j + 1$
 6. for n from $a.\text{used} - b$ to $a.\text{used} - 1$ do
 - 6.1 $a_n \leftarrow 0$
 7. $a.\text{used} \leftarrow a.\text{used} - b$
 8. Return.
-

Figure 4.11: Algorithm mp_rshd

Algorithm mp_rshd. This algorithm divides the input in place by the b 'th power of x . It is analogous to dividing by a β^b but much quicker since it does not require single precision division. This algorithm does not actually return an error code as it cannot fail.

If the input b is less than one the algorithm quickly returns without performing any work. If the **used** count is less than or equal to the shift count b then it will simply zero the input and return.

After the trivial cases of inputs have been handled the sliding window is setup. Much like the case of algorithm mp_lshd a sliding window that is b digits wide is used to copy the digits. Unlike mp_lshd the window slides in the opposite direction from the trailing to the leading digit. Also the digits are copied from the leading to the trailing edge.

Once the window copy is complete the upper digits must be zeroed and the **used** count decremented.

File: bn_mp_rshd.c

```
016
017  /* shift right a certain amount of digits */
018  void mp_rshd (mp_int * a, int b)
```

```

019  {
020      int      x;
021
022      /* if b <= 0 then ignore it */
023      if (b <= 0) {
024          return;
025      }
026
027      /* if b > used then simply zero it and return */
028      if (a->used <= b) {
029          mp_zero (a);
030          return;
031      }
032
033      {
034          mp_digit *bottom, *top;
035
036          /* shift the digits down */
037
038          /* bottom */
039          bottom = a->dp;
040
041          /* top [offset into digits] */
042          top = a->dp + b;
043
044          /* this is implemented as a sliding window where
045           * the window is b-digits long and digits from
046           * the top of the window are copied to the bottom
047           *
048           * e.g.
049
050           b-2 | b-1 | b0 | b1 | b2 | ... | bb | ---->
051                /\                | ---->
052                \-----/         ---->
053          */
054          for (x = 0; x < (a->used - b); x++) {
055              *bottom++ = *top++;
056          }
057
058          /* zero the top digits */
059          for (; x < a->used; x++) {
060              *bottom++ = 0;

```

```

061     }
062   }
063
064   /* remove excess digits */
065   a->used -= b;
066 }
067 #endif
068
069 /* ref:          tag: v1.0.1 */
070 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
071 /* commit time:  2017-08-29 22:27:36 +0200 */

```

The only noteworthy element of this routine is the lack of a return type since it cannot fail. Like `mp_lshd()` we form a sliding window except we copy in the other direction. After the window (line 59) we then zero the upper digits of the input to make sure the result is correct.

4.5 Powers of Two

Now that algorithms for moving single bits as well as whole digits exist algorithms for moving the “in between” distances are required. For example, to quickly multiply by 2^k for any k without using a full multiplier algorithm would prove useful. Instead of performing single shifts k times to achieve a multiplication by $2^{\pm k}$ a mixture of whole digit shifting and partial digit shifting is employed.

4.5.1 Multiplication by Power of Two

Algorithm **mp_mul_2d**.

Input. One mp_int a and an integer b

Output. $c \leftarrow a \cdot 2^b$.

1. $c \leftarrow a$. (*mp_copy*)
 2. If $c.alloc < c.used + \lfloor b/lg(\beta) \rfloor + 2$ then grow c accordingly.
 3. If the reallocation failed return(*MP_MEM*).
 4. If $b \geq lg(\beta)$ then
 - 4.1 $c \leftarrow c \cdot \beta^{\lfloor b/lg(\beta) \rfloor}$ (*mp_lshd*).
 - 4.2 If step 4.1 failed return(*MP_MEM*).
 5. $d \leftarrow b \pmod{lg(\beta)}$
 6. If $d \neq 0$ then do
 - 6.1 $mask \leftarrow 2^d$
 - 6.2 $r \leftarrow 0$
 - 6.3 for n from 0 to $c.used - 1$ do
 - 6.3.1 $rr \leftarrow c_n \gg (lg(\beta) - d) \pmod{mask}$
 - 6.3.2 $c_n \leftarrow (c_n \ll d) + r \pmod{\beta}$
 - 6.3.3 $r \leftarrow rr$
 - 6.4 If $r > 0$ then do
 - 6.4.1 $c_{c.used} \leftarrow r$
 - 6.4.2 $c.used \leftarrow c.used + 1$
 7. Return(*MP_OKAY*).
-

Figure 4.12: Algorithm mp_mul_2d

Algorithm mp_mul_2d. This algorithm multiplies a by 2^b and stores the result in c . The algorithm uses algorithm *mp_lshd* and a derivative of algorithm *mp_mul_2* to quickly compute the product.

First the algorithm will multiply a by $x^{\lfloor b/lg(\beta) \rfloor}$ which will ensure that the remainder multiplicand is less than β . For example, if $b = 37$ and $\beta = 2^{28}$ then this step will multiply by x leaving a multiplication by $2^{37-28} = 2^9$ left.

After the digits have been shifted appropriately at most $lg(\beta) - 1$ shifts are left to perform. Step 5 calculates the number of remaining shifts required. If it is non-zero a modified shift loop is used to calculate the remaining product. Essentially the loop is a generic version of algorithm *mp_mul_2* designed to handle any shift count in the range $1 \leq x < lg(\beta)$. The *mask* variable is used to extract the upper d bits to form the carry for the next iteration.

This algorithm is loosely measured as a $O(2n)$ algorithm which means that if the input is n -digits that it takes $2n$ “time” to complete. It is possible to optimize this algorithm down to a $O(n)$ algorithm at a cost of making the

algorithm slightly harder to follow.

File: bn_mp_mul_2d.c

```

016
017  /* shift left by a certain bit count */
018  int mp_mul_2d (mp_int * a, int b, mp_int * c)
019  {
020      mp_digit d;
021      int      res;
022
023      /* copy */
024      if (a != c) {
025          if ((res = mp_copy (a, c)) != MP_OKAY) {
026              return res;
027          }
028      }
029
030      if (c->alloc < (int)(c->used + (b / DIGIT_BIT) + 1)) {
031          if ((res = mp_grow (c, c->used + (b / DIGIT_BIT) + 1)) != MP_OKAY) {
032              return res;
033          }
034      }
035
036      /* shift by as many digits in the bit count */
037      if (b >= (int)DIGIT_BIT) {
038          if ((res = mp_lshd (c, b / DIGIT_BIT)) != MP_OKAY) {
039              return res;
040          }
041      }
042
043      /* shift any bit count < DIGIT_BIT */
044      d = (mp_digit) (b % DIGIT_BIT);
045      if (d != 0) {
046          mp_digit *tmpc, shift, mask, r, rr;
047          int x;
048
049          /* bitmask for carries */
050          mask = (((mp_digit)1) << d) - 1;
051
052          /* shift for msbs */
053          shift = DIGIT_BIT - d;
054
055          /* alias */

```

```

056     tmpc = c->dp;
057
058     /* carry */
059     r     = 0;
060     for (x = 0; x < c->used; x++) {
061         /* get the higher bits of the current word */
062         rr = (*tmpc >> shift) & mask;
063
064         /* shift the current word and OR in the carry */
065         *tmpc = ((*tmpc << d) | r) & MP_MASK;
066         ++tmpc;
067
068         /* set the carry to the carry bits of the current word */
069         r = rr;
070     }
071
072     /* set final carry */
073     if (r != 0) {
074         c->dp[(c->used)++] = r;
075     }
076 }
077 mp_clamp (c);
078 return MP_OKAY;
079 }
080 #endif
081
082 /* ref:          tag: v1.0.1 */
083 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
084 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The shifting is performed in-place which means the first step (line 24) is to copy the input to the destination. We avoid calling `mp_copy()` by making sure the `mp_ints` are different. The destination then has to be grown (line 31) to accomodate the result.

If the shift count b is larger than $lg(\beta)$ then a call to `mp_lshd()` is used to handle all of the multiples of $lg(\beta)$. Leaving only a remaining shift of $lg(\beta) - 1$ or fewer bits left. Inside the actual shift loop (lines 45 to 76) we make use of pre-computed values *shift* and *mask*. These are used to extract the carry bit(s) to pass into the next iteration of the loop. The *r* and *rr* variables form a chain between consecutive iterations to propagate the carry.

4.5.2 Division by Power of Two

Algorithm **mp_div_2d**.

Input. One mp_int a and an integer b

Output. $c \leftarrow \lfloor a/2^b \rfloor, d \leftarrow a \pmod{2^b}$.

1. If $b \leq 0$ then do
 - 1.1 $c \leftarrow a$ (*mp_copy*)
 - 1.2 $d \leftarrow 0$ (*mp_zero*)
 - 1.3 Return(*MP_OKAY*).
 2. $c \leftarrow a$
 3. $d \leftarrow a \pmod{2^b}$ (*mp_mod_2d*)
 4. If $b \geq \lg(\beta)$ then do
 - 4.1 $c \leftarrow \lfloor c/\beta^{\lfloor b/\lg(\beta) \rfloor} \rfloor$ (*mp_rshd*).
 5. $k \leftarrow b \pmod{\lg(\beta)}$
 6. If $k \neq 0$ then do
 - 6.1 $mask \leftarrow 2^k$
 - 6.2 $r \leftarrow 0$
 - 6.3 for n from $c.used - 1$ to 0 do
 - 6.3.1 $rr \leftarrow c_n \pmod{mask}$
 - 6.3.2 $c_n \leftarrow (c_n \gg k) + (r \ll (\lg(\beta) - k))$
 - 6.3.3 $r \leftarrow rr$
 7. Clamp excess digits of c . (*mp_clamp*)
 8. Return(*MP_OKAY*).
-

Figure 4.13: Algorithm mp_div_2d

Algorithm mp_div_2d. This algorithm will divide an input a by 2^b and produce the quotient and remainder. The algorithm is designed much like algorithm mp_mul_2d by first using whole digit shifts then single precision shifts. This algorithm will also produce the remainder of the division by using algorithm mp_mod_2d.

File: bn_mp_div_2d.c

```

016
017  /* shift right by a certain bit count (store quotient in c, optional remaind
    er in d) */
018  int mp_div_2d (mp_int * a, int b, mp_int * c, mp_int * d)
019  {
020      mp_digit D, r, rr;
021      int      x, res;
022
023      /* if the shift count is <= 0 then we do no work */

```



```
024     if (b <= 0) {
025         res = mp_copy (a, c);
026         if (d != NULL) {
027             mp_zero (d);
028         }
029         return res;
030     }
031
032     /* copy */
033     if ((res = mp_copy (a, c)) != MP_OKAY) {
034         return res;
035     }
036     /* 'a' should not be used after here - it might be the same as d */
037
038     /* get the remainder */
039     if (d != NULL) {
040         if ((res = mp_mod_2d (a, b, d)) != MP_OKAY) {
041             return res;
042         }
043     }
044
045     /* shift by as many digits in the bit count */
046     if (b >= (int)DIGIT_BIT) {
047         mp_rshd (c, b / DIGIT_BIT);
048     }
049
050     /* shift any bit count < DIGIT_BIT */
051     D = (mp_digit) (b % DIGIT_BIT);
052     if (D != 0) {
053         mp_digit *tmpc, mask, shift;
054
055         /* mask */
056         mask = (((mp_digit)1) << D) - 1;
057
058         /* shift for lsb */
059         shift = DIGIT_BIT - D;
060
061         /* alias */
062         tmpc = c->dp + (c->used - 1);
063
064         /* carry */
065         r = 0;
```

```

066     for (x = c->used - 1; x >= 0; x--) {
067         /* get the lower bits of this word in a temp */
068         rr = *tmpc & mask;
069
070         /* shift the current word and mix in the carry bits from the previous
word */
071         *tmpc = (*tmpc >> D) | (r << shift);
072         --tmpc;
073
074         /* set the carry to the carry bits of the current word found above */
075         r = rr;
076     }
077 }
078 mp_clamp (c);
079 return MP_OKAY;
080 }
081 #endif
082
083 /* ref:          tag: v1.0.1 */
084 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
085 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The implementation of algorithm `mp_div_2d` is slightly different than the algorithm specifies. The remainder d may be optionally ignored by passing **NULL** as the pointer to the `mp_int` variable. The temporary `mp_int` variable t is used to hold the result of the remainder operation until the end. This allows d and a to represent the same `mp_int` without modifying a before the quotient is obtained.

The remainder of the source code is essentially the same as the source code for `mp_mul_2d`. The only significant difference is the direction of the shifts.

4.5.3 Remainder of Division by Power of Two

The last algorithm in the series of polynomial basis power of two algorithms is calculating the remainder of division by 2^b . This algorithm benefits from the fact that in twos complement arithmetic $a \pmod{2^b}$ is the same as $a \text{ AND } 2^b - 1$.

Algorithm `mp_mod_2d`. This algorithm will quickly calculate the value of $a \pmod{2^b}$. First if b is less than or equal to zero the result is set to zero. If b is greater than the number of bits in a then it simply copies a to c and returns.

Algorithm **mp_mod_2d**.

Input. One mp_int a and an integer b

Output. $c \leftarrow a \pmod{2^b}$.

1. If $b \leq 0$ then do
 - 1.1 $c \leftarrow 0$ (*mp_zero*)
 - 1.2 Return(*MP_OKAY*).
 2. If $b > a.\text{used} \cdot \lg(\beta)$ then do
 - 2.1 $c \leftarrow a$ (*mp_copy*)
 - 2.2 Return the result of step 2.1.
 3. $c \leftarrow a$
 4. If step 3 failed return(*MP_MEM*).
 5. for n from $\lceil b/\lg(\beta) \rceil$ to $c.\text{used}$ do
 - 5.1 $c_n \leftarrow 0$
 6. $k \leftarrow b \pmod{\lg(\beta)}$
 7. $c_{\lfloor b/\lg(\beta) \rfloor} \leftarrow c_{\lfloor b/\lg(\beta) \rfloor} \pmod{2^k}$.
 8. Clamp excess digits of c . (*mp_clamp*)
 9. Return(*MP_OKAY*).
-

Figure 4.14: Algorithm mp_mod_2d

Otherwise, a is copied to b , leading digits are removed and the remaining leading digit is trimmed to the exact bit count.

File: bn_mp_mod_2d.c

```

016
017  /* calc a value mod 2**b */
018  int
019  mp_mod_2d (mp_int * a, int b, mp_int * c)
020  {
021      int      x, res;
022
023      /* if b is <= 0 then zero the int */
024      if (b <= 0) {
025          mp_zero (c);
026          return MP_OKAY;
027      }
028
029      /* if the modulus is larger than the value than return */
030      if (b >= (int) (a->used * DIGIT_BIT)) {
031          res = mp_copy (a, c);

```

```

032     return res;
033 }
034
035 /* copy */
036 if ((res = mp_copy (a, c)) != MP_OKAY) {
037     return res;
038 }
039
040 /* zero digits above the last digit of the modulus */
041 for (x = (b / DIGIT_BIT) + (((b % DIGIT_BIT) == 0) ? 0 : 1); x < c->used;
x++) {
042     c->dp[x] = 0;
043 }
044 /* clear the digit that is not completely outside/inside the modulus */
045 c->dp[b / DIGIT_BIT] &=
046     (mp_digit) (((mp_digit) 1) << (((mp_digit) b) % DIGIT_BIT)) - ((mp_digi
t) 1));
047 mp_clamp (c);
048 return MP_OKAY;
049 }
050 #endif
051
052 /* ref:          tag: v1.0.1 */
053 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
054 /* commit time: 2017-08-29 22:27:36 +0200 */

```

We first avoid cases of $b \leq 0$ by simply `mp_zero()`'ing the destination in such cases. Next if 2^b is larger than the input we just `mp_copy()` the input and return right away. After this point we know we must actually perform some work to produce the remainder.

Recalling that reducing modulo 2^k and a binary “and” with $2^k - 1$ are numerically equivalent we can quickly reduce the number. First we zero any digits above the last digit in 2^b (line 41). Next we reduce the leading digit of both (line 45) and then `mp_clamp()`.

Exercises

- [3] Devise an algorithm that performs $a \cdot 2^b$ for generic values of b in $O(n)$ time.
- [3] Devise an efficient algorithm to multiply by small low hamming weight values such as 3, 5 and 9. Extend it to handle all values upto 64 with a hamming weight less than three.
- [2] Modify the preceding algorithm to handle values of the form $2^k - 1$ as well.
- [3] Using only algorithms `mp_mul_2`, `mp_div_2` and `mp_add` create an algorithm to multiply two integers in roughly $O(2n^2)$ time for any n -bit input. Note that the time of addition is ignored in the calculation.
- [5] Improve the previous algorithm to have a working time of at most $O\left(2^{(k-1)}n + \left(\frac{2n^2}{k}\right)\right)$ for an appropriate choice of k . Again ignore the cost of addition.
- [2] Devise a chart to find optimal values of k for the previous problem for $n = 64 \dots 1024$ in steps of 64.
- [2] Using only algorithms `mp_abs` and `mp_sub` devise another method for calculating the result of a signed comparison.

Chapter 5

Multiplication and Squaring

5.1 The Multipliers

For most number theoretic problems including certain public key cryptographic algorithms, the “multipliers” form the most important subset of algorithms of any multiple precision integer package. The set of multiplier algorithms include integer multiplication, squaring and modular reduction where in each of the algorithms single precision multiplication is the dominant operation performed. This chapter will discuss integer multiplication and squaring, leaving modular reductions for the subsequent chapter.

The importance of the multiplier algorithms is for the most part driven by the fact that certain popular public key algorithms are based on modular exponentiation, that is computing $d \equiv a^b \pmod{c}$ for some arbitrary choice of a , b , c and d . During a modular exponentiation the majority¹ of the processor time is spent performing single precision multiplications.

For centuries general purpose multiplication has required a lengthy $O(n^2)$ process, whereby each digit of one multiplicand has to be multiplied against every digit of the other multiplicand. Traditional long-hand multiplication is based on this process; while the techniques can differ the overall algorithm used is essentially the same. Only “recently” have faster algorithms been studied. First Karatsuba multiplication was discovered in 1962. This algorithm can

¹Roughly speaking a modular exponentiation will spend about 40% of the time performing modular reductions, 35% of the time performing squaring and 25% of the time performing multiplications.

multiply two numbers with considerably fewer single precision multiplications when compared to the long-hand approach. This technique led to the discovery of polynomial basis algorithms (*good reference?*) and subsequently Fourier Transform based solutions.

5.2 Multiplication

5.2.1 The Baseline Multiplication

Computing the product of two integers in software can be achieved using a trivial adaptation of the standard $O(n^2)$ long-hand multiplication algorithm that school children are taught. The algorithm is considered an $O(n^2)$ algorithm since for two n -digit inputs n^2 single precision multiplications are required. More specifically for a m and n digit input $m \cdot n$ single precision multiplications are required. To simplify most discussions, it will be assumed that the inputs have comparable number of digits.

The “baseline multiplication” algorithm is designed to act as the “catch-all” algorithm, only to be used when the faster algorithms cannot be used. This algorithm does not use any particularly interesting optimizations and should ideally be avoided if possible. One important facet of this algorithm, is that it has been modified to only produce a certain amount of output digits as resolution. The importance of this modification will become evident during the discussion of Barrett modular reduction. Recall that for a n and m digit input the product will be at most $n + m$ digits. Therefore, this algorithm can be reduced to a full multiplier by having it produce $n + m$ digits of the product.

Recall from sub-section 4.2.2 the definition of γ as the number of bits in the type **mp_digit**. We shall now extend the variable set to include α which shall represent the number of bits in the type **mp_word**. This implies that $2^\alpha > 2 \cdot \beta^2$. The constant $\delta = 2^{\alpha - 2lg(\beta)}$ will represent the maximal weight of any column in a product (*see sub-section 5.2.2 for more information*).

Algorithm **s_mp_mul_digs**.

Input. mp_int a , mp_int b and an integer $digs$

Output. $c \leftarrow |a| \cdot |b| \pmod{\beta^{digs}}$.

1. If $\min(a.used, b.used) < \delta$ then do
 - 1.1 Calculate $c = |a| \cdot |b|$ by the Comba method (*see algorithm 5.5*).
 - 1.2 Return the result of step 1.1

Allocate and initialize a temporary mp_int.

2. Init t to be of size $digs$
3. If step 2 failed return(MP_MEM).
4. $t.used \leftarrow digs$

Compute the product.

5. for ix from 0 to $a.used - 1$ do
 - 5.1 $u \leftarrow 0$
 - 5.2 $pb \leftarrow \min(b.used, digs - ix)$
 - 5.3 If $pb < 1$ then goto step 6.
 - 5.4 for iy from 0 to $pb - 1$ do
 - 5.4.1 $\hat{r} \leftarrow t_{iy+ix} + a_{ix} \cdot b_{iy} + u$
 - 5.4.2 $t_{iy+ix} \leftarrow \hat{r} \pmod{\beta}$
 - 5.4.3 $u \leftarrow \lfloor \hat{r} / \beta \rfloor$
 - 5.5 if $ix + pb < digs$ then do
 - 5.5.1 $t_{ix+pb} \leftarrow u$
 6. Clamp excess digits of t .
 7. Swap c with t
 8. Clear t
 9. Return(MP_OKAY).
-

Figure 5.1: Algorithm s_mp_mul_digs

Algorithm s_mp_mul_digs. This algorithm computes the unsigned product of two inputs a and b , limited to an output precision of $digs$ digits. While it may seem a bit awkward to modify the function from its simple $O(n^2)$ description, the usefulness of partial multipliers will arise in a subsequent algorithm. The algorithm is loosely based on algorithm 14.12 from [2, pp. 595] and is similar to Algorithm M of Knuth [1, pp. 268]. Algorithm s_mp_mul_digs differs from these cited references since it can produce a variable output precision regardless of the precision of the inputs.

The first thing this algorithm checks for is whether a Comba multiplier can

be used instead. If the minimum digit count of either input is less than δ , then the Comba method may be used instead. After the Comba method is ruled out, the baseline algorithm begins. A temporary mp_int variable t is used to hold the intermediate result of the product. This allows the algorithm to be used to compute products when either $a = c$ or $b = c$ without overwriting the inputs.

All of step 5 is the infamous $O(n^2)$ multiplication loop slightly modified to only produce upto $digs$ digits of output. The pb variable is given the count of digits to read from b inside the nested loop. If $pb \leq 1$ then no more output digits can be produced and the algorithm will exit the loop. The best way to think of the loops are as a series of $pb \times 1$ multiplications. That is, in each pass of the innermost loop a_{ix} is multiplied against b and the result is added (*with an appropriate shift*) to t .

For example, consider multiplying 576 by 241. That is equivalent to computing $10^0(1)(576) + 10^1(4)(576) + 10^2(2)(576)$ which is best visualized in the following table.

| | | | | | | |
|----------|---|---|---|---|---|--|
| | | | 5 | 7 | 6 | |
| \times | | | 2 | 4 | 1 | |
| | | | 5 | 7 | 6 | $10^0(1)(576)$ |
| | 2 | 3 | 6 | 1 | 6 | $10^1(4)(576) + 10^0(1)(576)$ |
| 1 | 3 | 8 | 8 | 1 | 6 | $10^2(2)(576) + 10^1(4)(576) + 10^0(1)(576)$ |

Figure 5.2: Long-Hand Multiplication Diagram

Each row of the product is added to the result after being shifted to the left (*multiplied by a power of the radix*) by the appropriate count. That is in pass ix of the inner loop the product is added starting at the ix 'th digit of the result.

Step 5.4.1 introduces the hat symbol (*e.g.* \hat{r}) which represents a double precision variable. The multiplication on that step is assumed to be a double wide output single precision multiplication. That is, two single precision variables are multiplied to produce a double precision result. The step is somewhat optimized from a long-hand multiplication algorithm because the carry from the addition in step 5.4.1 is propagated through the nested loop. If the carry was not propagated immediately it would overflow the single precision digit t_{ix+iy} and the result would be lost.

At step 5.5 the nested loop is finished and any carry that was left over should be forwarded. The carry does not have to be added to the $ix + pb$ 'th digit since

that digit is assumed to be zero at this point. However, if $ix + pb \geq digs$ the carry is not set as it would make the result exceed the precision requested.

File: bn_s_mp_mul_digs.c

```

016
017 /* multiplies |a| * |b| and only computes upto digs digits of result
018  * HAC pp. 595, Algorithm 14.12 Modified so you can control how
019  * many digits of output are created.
020  */
021 int s_mp_mul_digs (mp_int * a, mp_int * b, mp_int * c, int digs)
022 {
023     mp_int t;
024     int res, pa, pb, ix, iy;
025     mp_digit u;
026     mp_word r;
027     mp_digit tmpx, *tmpt, *tmpy;
028
029     /* can we use the fast multiplier? */
030     if (((digs) < MP_WARRAY) &&
031         (MIN (a->used, b->used) <
032          (1 << ((CHAR_BIT * sizeof(mp_word)) - (2 * DIGIT_BIT))))) {
033         return fast_s_mp_mul_digs (a, b, c, digs);
034     }
035
036     if ((res = mp_init_size (&t, digs)) != MP_OKAY) {
037         return res;
038     }
039     t.used = digs;
040
041     /* compute the digits of the product directly */
042     pa = a->used;
043     for (ix = 0; ix < pa; ix++) {
044         /* set the carry to zero */
045         u = 0;
046
047         /* limit ourselves to making digs digits of output */
048         pb = MIN (b->used, digs - ix);
049
050         /* setup some aliases */
051         /* copy of the digit from a used within the nested loop */
052         tmpx = a->dp[ix];
053
054         /* an alias for the destination shifted ix places */

```

```

055     tmp = t.dp + ix;
056
057     /* an alias for the digits of b */
058     tmpy = b->dp;
059
060     /* compute the columns of the output and propagate the carry */
061     for (iy = 0; iy < pb; iy++) {
062         /* compute the column as a mp_word */
063         r = (mp_word)*tmp +
064             ((mp_word)tmpx * (mp_word)*tmpy++) +
065             (mp_word)u;
066
067         /* the new column is the lower part of the result */
068         *tmp++ = (mp_digit) (r & ((mp_word) MP_MASK));
069
070         /* get the carry word from the result */
071         u = (mp_digit) (r >> ((mp_word) DIGIT_BIT));
072     }
073     /* set carry if it is placed below digs */
074     if ((ix + iy) < digs) {
075         *tmp = u;
076     }
077 }
078
079 mp_clamp (&t);
080 mp_exch (&t, c);
081
082 mp_clear (&t);
083 return MP_OKAY;
084 }
085 #endif
086
087 /* ref:          tag: v1.0.1 */
088 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
089 /* commit time: 2017-08-29 22:27:36 +0200 */

```

First we determine (line 30) if the Comba method can be used first since it's faster. The conditions for using the Comba routine are that $\min(a.\text{used}, b.\text{used}) < \delta$ and the number of digits of output is less than **MP_WARRAY**. This new constant is used to control the stack usage in the Comba routines. By default it is set to δ but can be reduced when memory is at a premium.

If we cannot use the Comba method we proceed to setup the baseline routine.

We allocate the destination mp_int t (line 36) to the exact size of the output to avoid further re-allocations. At this point we now begin the $O(n^2)$ loop.

This implementation of multiplication has the caveat that it can be trimmed to only produce a variable number of digits as output. In each iteration of the outer loop the pb variable is set (line 48) to the maximum number of inner loop iterations.

Inside the inner loop we calculate \hat{r} as the mp_word product of the two mp_digits and the addition of the carry from the previous iteration. A particularly important observation is that most modern optimizing C compilers (GCC for instance) can recognize that a $N \times N \rightarrow 2N$ multiplication is all that is required for the product. In x86 terms for example, this means using the MUL instruction.

Each digit of the product is stored in turn (line 68) and the carry propagated (line 71) to the next iteration.

5.2.2 Faster Multiplication by the “Comba” Method

One of the huge drawbacks of the “baseline” algorithms is that at the $O(n^2)$ level the carry must be computed and propagated upwards. This makes the nested loop very sequential and hard to unroll and implement in parallel. The “Comba” [4] method is named after little known (*in cryptographic venues*) Paul G. Comba who described a method of implementing fast multipliers that do not require nested carry fixup operations. As an interesting aside it seems that Paul Barrett describes a similar technique in his 1986 paper [7] written five years before.

At the heart of the Comba technique is once again the long-hand algorithm. Except in this case a slight twist is placed on how the columns of the result are produced. In the standard long-hand algorithm rows of products are produced then added together to form the final result. In the baseline algorithm the columns are added together after each iteration to get the result instantaneously.

In the Comba algorithm the columns of the result are produced entirely independently of each other. That is at the $O(n^2)$ level a simple multiplication and addition step is performed. The carries of the columns are propagated after the nested loop to reduce the amount of work required. Succintly the first step of the algorithm is to compute the product vector \vec{x} as follows.

$$\vec{x}_n = \sum_{i+j=n} a_i b_j, \forall n \in \{0, 1, 2, \dots, i+j\} \quad (5.1)$$

Where \vec{x}_n is the n' th column of the output vector. Consider the following example which computes the vector \vec{x} for the multiplication of 576 and 241.

| | | | | | |
|------------------|-----------------------|-----------------------|----------------------|-----------------|--------------|
| | | 5 | 7 | 6 | First Input |
| \times | | 2 | 4 | 1 | Second Input |
| | | $1 \cdot 5 = 5$ | $1 \cdot 7 = 7$ | $1 \cdot 6 = 6$ | First pass |
| | $4 \cdot 5 = 20$ | $4 \cdot 7 + 5 = 33$ | $4 \cdot 6 + 7 = 31$ | 6 | Second pass |
| $2 \cdot 5 = 10$ | $2 \cdot 7 + 20 = 34$ | $2 \cdot 6 + 33 = 45$ | 31 | 6 | Third pass |
| 10 | 34 | 45 | 31 | 6 | Final Result |

Figure 5.3: Comba Multiplication Diagram

At this point the vector $x = \langle 10, 34, 45, 31, 6 \rangle$ is the result of the first step of the Comba multiplier. Now the columns must be fixed by propagating the carry upwards. The resultant vector will have one extra dimension over the input vector which is congruent to adding a leading zero digit.

Algorithm **Comba Fixup**.

Input. Vector \vec{x} of dimension k

Output. Vector \vec{x} such that the carries have been propagated.

1. for n from 0 to $k - 1$ do
 - 1.1 $\vec{x}_{n+1} \leftarrow \vec{x}_{n+1} + \lfloor \vec{x}_n / \beta \rfloor$
 - 1.2 $\vec{x}_n \leftarrow \vec{x}_n \pmod{\beta}$
 2. Return(\vec{x}).
-

Figure 5.4: Algorithm Comba Fixup

With that algorithm and $k = 5$ and $\beta = 10$ the following vector is produced $\vec{x} = \langle 1, 3, 8, 8, 1, 6 \rangle$. In this case $241 \cdot 576$ is in fact 138816 and the procedure succeeded. If the algorithm is correct and as will be demonstrated shortly more efficient than the baseline algorithm why not simply always use this algorithm?

Column Weight.

At the nested $O(n^2)$ level the Comba method adds the product of two single precision variables to each column of the output independently. A serious obstacle is if the carry is lost, due to lack of precision before the algorithm has a chance to fix the carries. For example, in the multiplication of two three-digit numbers the third column of output will be the sum of three single precision multiplications. If the precision of the accumulator for the output digits is less than $3 \cdot (\beta - 1)^2$ then an overflow can occur and the carry information will be lost.

For any m and n digit inputs the maximum weight of any column is $\min(m, n)$ which is fairly obvious.

The maximum number of terms in any column of a product is known as the “column weight” and strictly governs when the algorithm can be used. Recall from earlier that a double precision type has α bits of resolution and a single precision digit has $lg(\beta)$ bits of precision. Given these two quantities we must not violate the following

$$k \cdot (\beta - 1)^2 < 2^\alpha \quad (5.2)$$

Which reduces to

$$k \cdot (\beta^2 - 2\beta + 1) < 2^\alpha \quad (5.3)$$

Let $\rho = lg(\beta)$ represent the number of bits in a single precision digit. By further re-arrangement of the equation the final solution is found.

$$k < \frac{2^\alpha}{(2^{2\rho} - 2^{\rho+1} + 1)} \quad (5.4)$$

The defaults for LibTomMath are $\beta = 2^{28}$ and $\alpha = 2^{64}$ which means that k is bounded by $k < 257$. In this configuration the smaller input may not have more than 256 digits if the Comba method is to be used. This is quite satisfactory for most applications since 256 digits would allow for numbers in the range of $0 \leq x < 2^{7168}$ which, is much larger than most public key cryptographic algorithms require.

Algorithm **fast_s_mp_mul_digs**.

Input. mp_int a , mp_int b and an integer $digs$

Output. $c \leftarrow |a| \cdot |b| \pmod{\beta^{digs}}$.

Place an array of **MP_WARRAY** single precision digits named W on the stack.

1. If $c.alloc < digs$ then grow c to $digs$ digits. (*mp-grow*)
2. If step 1 failed return(*MP_MEM*).

3. $pa \leftarrow \text{MIN}(digs, a.used + b.used)$

4. $\hat{W} \leftarrow 0$

5. for ix from 0 to $pa - 1$ do

5.1 $ty \leftarrow \text{MIN}(b.used - 1, ix)$

5.2 $tx \leftarrow ix - ty$

5.3 $iy \leftarrow \text{MIN}(a.used - tx, ty + 1)$

5.4 for iz from 0 to $iy - 1$ do

5.4.1 $\hat{W} \leftarrow \hat{W} + a_{tx+iy}b_{ty-iz}$

5.5 $W_{ix} \leftarrow \hat{W} \pmod{\beta}$

5.6 $\hat{W} \leftarrow \lfloor \hat{W} / \beta \rfloor$

6. $oldused \leftarrow c.used$

7. $c.used \leftarrow digs$

8. for ix from 0 to pa do

8.1 $c_{ix} \leftarrow W_{ix}$

9. for ix from $pa + 1$ to $oldused - 1$ do

9.1 $c_{ix} \leftarrow 0$

10. Clamp c .

11. Return *MP_OKAY*.

Figure 5.5: Algorithm fast_s_mp_mul_digs

Algorithm fast_s_mp_mul_digs. This algorithm performs the unsigned multiplication of a and b using the Comba method limited to $digs$ digits of precision.

The outer loop of this algorithm is more complicated than that of the baseline multiplier. This is because on the inside of the loop we want to produce one column per pass. This allows the accumulator \hat{W} to be placed in CPU registers and reduce the memory bandwidth to two **mp_digit** reads per iteration.

The ty variable is set to the minimum count of ix or the number of digits in

b. That way if *a* has more digits than *b* this will be limited to $b.used - 1$. The *tx* variable is set to the distance past *b.used* the variable *ix* is. This is used for the immediately subsequent statement where we find *iy*.

The variable *iy* is the minimum digits we can read from either *a* or *b* before running out. Computing one column at a time means we have to scan one integer upwards and the other downwards. *a* starts at *tx* and *b* starts at *ty*. In each pass we are producing the *ix*'th output column and we note that $tx + ty = ix$. As we move *tx* upwards we have to move *ty* downwards so the equality remains valid. The *iy* variable is the number of iterations until $tx \geq a.used$ or $ty < 0$ occurs.

After every inner pass we store the lower half of the accumulator into W_{ix} and then propagate the carry of the accumulator into the next round by dividing \hat{W} by β .

To measure the benefits of the Comba method over the baseline method consider the number of operations that are required. If the cost in terms of time of a multiply and addition is p and the cost of a carry propagation is q then a baseline multiplication would require $O((p + q)n^2)$ time to multiply two n -digit numbers. The Comba method requires only $O(pn^2 + qn)$ time, however in practice, the speed increase is actually much more. With $O(n)$ space the algorithm can be reduced to $O(pn + qn)$ time by implementing the n multiply and addition operations in the nested loop in parallel.

File: bn_fast_s_mp_mul_digs.c

```

016
017  /* Fast (comba) multiplier
018  *
019  * This is the fast column-array [comba] multiplier. It is
020  * designed to compute the columns of the product first
021  * then handle the carries afterwards. This has the effect
022  * of making the nested loops that compute the columns very
023  * simple and schedulable on super-scalar processors.
024  *
025  * This has been modified to produce a variable number of
026  * digits of output so if say only a half-product is required
027  * you don't have to compute the upper half (a feature
028  * required for fast Barrett reduction).
029  *
030  * Based on Algorithm 14.12 on pp.595 of HAC.
031  *
032  */
033  int fast_s_mp_mul_digs (mp_int * a, mp_int * b, mp_int * c, int digs)

```

```

034  {
035      int      olduse, res, pa, ix, iz;
036      mp_digit W[MP_WARRAY];
037      mp_word  _W;
038
039      /* grow the destination as required */
040      if (c->alloc < digs) {
041          if ((res = mp_grow (c, digs)) != MP_OKAY) {
042              return res;
043          }
044      }
045
046      /* number of output digits to produce */
047      pa = MIN(digs, a->used + b->used);
048
049      /* clear the carry */
050      _W = 0;
051      for (ix = 0; ix < pa; ix++) {
052          int      tx, ty;
053          int      iy;
054          mp_digit *tmpx, *tmpy;
055
056          /* get offsets into the two bignums */
057          ty = MIN(b->used-1, ix);
058          tx = ix - ty;
059
060          /* setup temp aliases */
061          tmpx = a->dp + tx;
062          tmpy = b->dp + ty;
063
064          /* this is the number of times the loop will iterate, essentially
065             while (tx++ < a->used && ty-- >= 0) { ... }
066             */
067          iy = MIN(a->used-tx, ty+1);
068
069          /* execute loop */
070          for (iz = 0; iz < iy; ++iz) {
071              _W += ((mp_word)*tmpx++)*((mp_word)*tmpy--);
072          }
073      }
074
075      /* store term */

```

```

076         W[ix] = ((mp_digit)_W) & MP_MASK;
077
078         /* make next carry */
079         _W = _W >> ((mp_word)DIGIT_BIT);
080     }
081
082     /* setup dest */
083     olduse = c->used;
084     c->used = pa;
085
086     {
087         mp_digit *tmpc;
088         tmpc = c->dp;
089         for (ix = 0; ix < (pa + 1); ix++) {
090             /* now extract the previous digit [below the carry] */
091             *tmpc++ = W[ix];
092         }
093
094         /* clear unused digits [that existed in the old copy of c] */
095         for (; ix < olduse; ix++) {
096             *tmpc++ = 0;
097         }
098     }
099     mp_clamp(c);
100     return MP_OKAY;
101 }
102 #endif
103
104 /* ref:          tag: v1.0.1 */
105 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
106 /* commit time: 2017-08-29 22:27:36 +0200 */

```

As per the pseudo-code we first calculate pa (line 47) as the number of digits to output. Next we begin the outer loop to produce the individual columns of the product. We use the two aliases *tmpx* and *tmpy* (lines 61, 62) to point inside the two multiplicands quickly.

The inner loop (lines 70 to 73) of this implementation is where the tradeoff come into play. Originally this comba implementation was “row-major” which means it adds to each of the columns in each pass. After the outer loop it would then fix the carries. This was very fast except it had an annoying drawback. You had to read a `mp_word` and two `mp_digits` and write one `mp_word` per iteration. On processors such as the Athlon XP and P4 this did not matter

much since the cache bandwidth is very high and it can keep the ALU fed with data. It did, however, matter on older and embedded cpus where cache is often slower and also often doesn't exist. This new algorithm only performs two reads per iteration under the assumption that the compiler has aliased \hat{W} to a CPU register.

After the inner loop we store the current accumulator in W and shift \hat{W} (lines 76, 79) to forward it as a carry for the next pass. After the outer loop we use the final carry (line 76) as the last digit of the product.

5.2.3 Polynomial Basis Multiplication

To break the $O(n^2)$ barrier in multiplication requires a completely different look at integer multiplication. In the following algorithms the use of polynomial basis representation for two integers a and b as $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{i=0}^n b_i x^i$ respectively, is required. In this system both $f(x)$ and $g(x)$ have $n+1$ terms and are of the n 'th degree.

The product $a \cdot b \equiv f(x)g(x)$ is the polynomial $W(x) = \sum_{i=0}^{2n} w_i x^i$. The coefficients w_i will directly yield the desired product when β is substituted for x . The direct solution to solve for the $2n+1$ coefficients requires $O(n^2)$ time and would in practice be slower than the Comba technique.

However, numerical analysis theory indicates that only $2n+1$ distinct points in $W(x)$ are required to determine the values of the $2n+1$ unknown coefficients. This means by finding $\zeta_y = W(y)$ for $2n+1$ small values of y the coefficients of $W(x)$ can be found with Gaussian elimination. This technique is also occasionally referred to as the *interpolation technique* (*references please...*) since in effect an interpolation based on $2n+1$ points will yield a polynomial equivalent to $W(x)$.

The coefficients of the polynomial $W(x)$ are unknown which makes finding $W(y)$ for any value of y impossible. However, since $W(x) = f(x)g(x)$ the equivalent $\zeta_y = f(y)g(y)$ can be used in its place. The benefit of this technique stems from the fact that $f(y)$ and $g(y)$ are much smaller than either a or b respectively. As a result finding the $2n+1$ relations required by multiplying $f(y)g(y)$ involves multiplying integers that are much smaller than either of the inputs.

When picking points to gather relations there are always three obvious points to choose, $y = 0, 1$ and ∞ . The ζ_0 term is simply the product $W(0) = w_0 = a_0 \cdot b_0$. The ζ_1 term is the product $W(1) = (\sum_{i=0}^n a_i) (\sum_{i=0}^n b_i)$. The third point ζ_∞ is less obvious but rather simple to explain. The $2n+1$ 'th coefficient of $W(x)$ is numerically equivalent to the most significant column in an integer

| Split into n Parts | Exponent | Notes |
|----------------------|-------------|-----------------------------------|
| 2 | 1.584962501 | This is Karatsuba Multiplication. |
| 3 | 1.464973520 | This is Toom-Cook Multiplication. |
| 4 | 1.403677461 | |
| 5 | 1.365212389 | |
| 10 | 1.278753601 | |
| 100 | 1.149426538 | |
| 1000 | 1.100270931 | |
| 10000 | 1.075252070 | |

Figure 5.6: Asymptotic Running Time of Polynomial Basis Multiplication

multiplication. The point at ∞ is used symbolically to represent the most significant column, that is $W(\infty) = w_{2n} = a_n b_n$. Note that the points at $y = 0$ and ∞ yield the coefficients w_0 and w_{2n} directly.

If more points are required they should be of small values and powers of two such as 2^q and the related *mirror points* $(2^q)^{2n} \cdot \zeta_{2^{-q}}$ for small values of q . The term “mirror point” stems from the fact that $(2^q)^{2n} \cdot \zeta_{2^{-q}}$ can be calculated in the exact opposite fashion as ζ_{2^q} . For example, when $n = 2$ and $q = 1$ then following two equations are equivalent to the point ζ_2 and its mirror.

$$\begin{aligned} \zeta_2 &= f(2)g(2) = (4a_2 + 2a_1 + a_0)(4b_2 + 2b_1 + b_0) \\ 16 \cdot \zeta_{\frac{1}{2}} &= 4f\left(\frac{1}{2}\right) \cdot 4g\left(\frac{1}{2}\right) = (a_2 + 2a_1 + 4a_0)(b_2 + 2b_1 + 4b_0) \end{aligned} \quad (5.5)$$

Using such points will allow the values of $f(y)$ and $g(y)$ to be independently calculated using only left shifts. For example, when $n = 2$ the polynomial $f(2^q)$ is equal to $2^q((2^q a_2) + a_1) + a_0$. This technique of polynomial representation is known as Horner’s method.

As a general rule of the algorithm when the inputs are split into n parts each there are $2n - 1$ multiplications. Each multiplication is of multiplicands that have n times fewer digits than the inputs. The asymptotic running time of this algorithm is $O(k^{lg_n(2n-1)})$ for k digit inputs (*assuming they have the same number of digits*). Figure 5.6 summarizes the exponents for various values of n .

At first it may seem like a good idea to choose $n = 1000$ since the exponent is approximately 1.1. However, the overhead of solving for the 2001 terms of

$W(x)$ will certainly consume any savings the algorithm could offer for all but exceedingly large numbers.

Cutoff Point

The polynomial basis multiplication algorithms all require fewer single precision multiplications than a straight Comba approach. However, the algorithms incur an overhead (*at the $O(n)$ work level*) since they require a system of equations to be solved. This makes the polynomial basis approach more costly to use with small inputs.

Let m represent the number of digits in the multiplicands (*assume both multiplicands have the same number of digits*). There exists a point y such that when $m < y$ the polynomial basis algorithms are more costly than Comba, when $m = y$ they are roughly the same cost and when $m > y$ the Comba methods are slower than the polynomial basis algorithms.

The exact location of y depends on several key architectural elements of the computer platform in question.

1. The ratio of clock cycles for single precision multiplication versus other simpler operations such as addition, shifting, etc. For example on the AMD Athlon the ratio is roughly 17 : 1 while on the Intel P4 it is 29 : 1. The higher the ratio in favour of multiplication the lower the cutoff point y will be.
2. The complexity of the linear system of equations (*for the coefficients of $W(x)$*) is. Generally speaking as the number of splits grows the complexity grows substantially. Ideally solving the system will only involve addition, subtraction and shifting of integers. This directly reflects on the ratio previous mentioned.
3. To a lesser extent memory bandwidth and function call overheads. Provided the values are in the processor cache this is less of an influence over the cutoff point.

A clean cutoff point separation occurs when a point y is found such that all of the cutoff point conditions are met. For example, if the point is too low then there will be values of m such that $m > y$ and the Comba method is still faster. Finding the cutoff points is fairly simple when a high resolution timer is available.

5.2.4 Karatsuba Multiplication

Karatsuba [5] multiplication when originally proposed in 1962 was among the first set of algorithms to break the $O(n^2)$ barrier for general purpose multiplication. Given two polynomial basis representations $f(x) = ax + b$ and $g(x) = cx + d$, Karatsuba proved with light algebra [6] that the following polynomial is equivalent to multiplication of the two integers the polynomials represent.

$$f(x) \cdot g(x) = acx^2 + ((a + b)(c + d) - (ac + bd))x + bd \quad (5.6)$$

Using the observation that ac and bd could be re-used only three half sized multiplications would be required to produce the product. Applying this algorithm recursively, the work factor becomes $O(n^{\lg(3)})$ which is substantially better than the work factor $O(n^2)$ of the Comba technique. It turns out what Karatsuba did not know or at least did not publish was that this is simply polynomial basis multiplication with the points ζ_0 , ζ_∞ and ζ_1 . Consider the resultant system of equations.

$$\begin{array}{rclclcl} \zeta_0 & = & & & & w_0 \\ \zeta_1 & = & w_2 & + & w_1 & + & w_0 \\ \zeta_\infty & = & w_2 & & & & \end{array}$$

By adding the first and last equation to the equation in the middle the term w_1 can be isolated and all three coefficients solved for. The simplicity of this system of equations has made Karatsuba fairly popular. In fact the cutoff point is often fairly low² making it an ideal algorithm to speed up certain public key cryptosystems such as RSA and Diffie-Hellman.

²With LibTomMath 0.18 it is 70 and 109 digits for the Intel P4 and AMD Athlon respectively.

Algorithm **mp_karatsuba_mul**.

Input. mp_int a and mp_int b

Output. $c \leftarrow |a| \cdot |b|$

1. Init the following mp_int variables: $x0, x1, y0, y1, t1, x0y0, x1y1$.
2. If step 2 failed then return(MP_MEM).

Split the input. e.g. $a = x1 \cdot \beta^B + x0$

3. $B \leftarrow \min(a.used, b.used)/2$
4. $x0 \leftarrow a \pmod{\beta^B}$ (mp_mod_2d)
5. $y0 \leftarrow b \pmod{\beta^B}$
6. $x1 \leftarrow \lfloor a/\beta^B \rfloor$ (mp_rshd)
7. $y1 \leftarrow \lfloor b/\beta^B \rfloor$

Calculate the three products.

8. $x0y0 \leftarrow x0 \cdot y0$ (mp_mul)
9. $x1y1 \leftarrow x1 \cdot y1$
10. $t1 \leftarrow x1 + x0$ (mp_add)
11. $x0 \leftarrow y1 + y0$
12. $t1 \leftarrow t1 \cdot x0$

Calculate the middle term.

13. $x0 \leftarrow x0y0 + x1y1$
14. $t1 \leftarrow t1 - x0$ (s_mp_sub)

Calculate the final product.

15. $t1 \leftarrow t1 \cdot \beta^B$ (mp_lshd)
 16. $x1y1 \leftarrow x1y1 \cdot \beta^{2B}$
 17. $t1 \leftarrow x0y0 + t1$
 18. $c \leftarrow t1 + x1y1$
 19. Clear all of the temporary variables.
 20. Return(MP_OKAY).
-

Figure 5.7: Algorithm mp_karatsuba_mul

Algorithm mp_karatsuba_mul. This algorithm computes the unsigned product of two inputs using the Karatsuba multiplication algorithm. It is loosely based on the description from Knuth [1, pp. 294-295].

In order to split the two inputs into their respective halves, a suitable *radix point* must be chosen. The radix point chosen must be used for both of the inputs meaning that it must be smaller than the smallest input. Step 3 chooses

the radix point B as half of the smallest input **used** count. After the radix point is chosen the inputs are split into lower and upper halves. Step 4 and 5 compute the lower halves. Step 6 and 7 compute the upper halves.

After the halves have been computed the three intermediate half-size products must be computed. Step 8 and 9 compute the trivial products $x_0 \cdot y_0$ and $x_1 \cdot y_1$. The mp_int x_0 is used as a temporary variable after $x_1 + x_0$ has been computed. By using x_0 instead of an additional temporary variable, the algorithm can avoid an addition memory allocation operation.

The remaining steps 13 through 18 compute the Karatsuba polynomial through a variety of digit shifting and addition operations.

File: bn_mp_karatsuba_mul.c

```

016
017  /* c = |a| * |b| using Karatsuba Multiplication using
018     * three half size multiplications
019     *
020     * Let B represent the radix [e.g. 2**DIGIT_BIT] and
021     * let n represent half of the number of digits in
022     * the min(a,b)
023     *
024     * a = a1 * B**n + a0
025     * b = b1 * B**n + b0
026     *
027     * Then, a * b =>
028     a1b1 * B**2n + ((a1 + a0)(b1 + b0) - (a0b0 + a1b1)) * B + a0b0
029     *
030     * Note that a1b1 and a0b0 are used twice and only need to be
031     * computed once. So in total three half size (half # of
032     * digit) multiplications are performed, a0b0, a1b1 and
033     * (a1+b1)(a0+b0)
034     *
035     * Note that a multiplication of half the digits requires
036     * 1/4th the number of single precision multiplications so in
037     * total after one call 25% of the single precision multiplications
038     * are saved. Note also that the call to mp_mul can end up back
039     * in this function if the a0, a1, b0, or b1 are above the threshold.
040     * This is known as divide-and-conquer and leads to the famous
041     *  $O(N \lg(3))$  or  $O(N^{1.584})$  work which is asymptotically lower than
042     * the standard  $O(N^2)$  that the baseline/comba methods use.
043     * Generally though the overhead of this method doesn't pay off
044     * until a certain size ( $N \sim 80$ ) is reached.
045     */

```

```

046 int mp_karatsuba_mul (mp_int * a, mp_int * b, mp_int * c)
047 {
048     mp_int  x0, x1, y0, y1, t1, x0y0, x1y1;
049     int      B, err;
050
051     /* default the return code to an error */
052     err = MP_MEM;
053
054     /* min # of digits */
055     B = MIN (a->used, b->used);
056
057     /* now divide in two */
058     B = B >> 1;
059
060     /* init copy all the temps */
061     if (mp_init_size (&x0, B) != MP_OKAY)
062         goto ERR;
063     if (mp_init_size (&x1, a->used - B) != MP_OKAY)
064         goto X0;
065     if (mp_init_size (&y0, B) != MP_OKAY)
066         goto X1;
067     if (mp_init_size (&y1, b->used - B) != MP_OKAY)
068         goto Y0;
069
070     /* init temps */
071     if (mp_init_size (&t1, B * 2) != MP_OKAY)
072         goto Y1;
073     if (mp_init_size (&x0y0, B * 2) != MP_OKAY)
074         goto T1;
075     if (mp_init_size (&x1y1, B * 2) != MP_OKAY)
076         goto X0Y0;
077
078     /* now shift the digits */
079     x0.used = y0.used = B;
080     x1.used = a->used - B;
081     y1.used = b->used - B;
082
083     {
084         int x;
085         mp_digit *tmpa, *tmpb, *tmpx, *tmpy;
086
087         /* we copy the digits directly instead of using higher level functions

```

```

088      * since we also need to shift the digits
089      */
090      tmpa = a->dp;
091      tmpb = b->dp;
092
093      tmpx = x0.dp;
094      tmpy = y0.dp;
095      for (x = 0; x < B; x++) {
096          *tmpx++ = *tmpa++;
097          *tmpy++ = *tmpb++;
098      }
099
100      tmpx = x1.dp;
101      for (x = B; x < a->used; x++) {
102          *tmpx++ = *tmpa++;
103      }
104
105      tmpy = y1.dp;
106      for (x = B; x < b->used; x++) {
107          *tmpy++ = *tmpb++;
108      }
109  }
110
111  /* only need to clamp the lower words since by definition the
112   * upper words x1/y1 must have a known number of digits
113   */
114  mp_clamp (&x0);
115  mp_clamp (&y0);
116
117  /* now calc the products x0y0 and x1y1 */
118  /* after this x0 is no longer required, free temp [x0==t2]! */
119  if (mp_mul (&x0, &y0, &x0y0) != MP_OKAY)
120      goto X1Y1;          /* x0y0 = x0*y0 */
121  if (mp_mul (&x1, &y1, &x1y1) != MP_OKAY)
122      goto X1Y1;          /* x1y1 = x1*y1 */
123
124  /* now calc x1+x0 and y1+y0 */
125  if (s_mp_add (&x1, &x0, &t1) != MP_OKAY)
126      goto X1Y1;          /* t1 = x1 - x0 */
127  if (s_mp_add (&y1, &y0, &x0) != MP_OKAY)
128      goto X1Y1;          /* t2 = y1 - y0 */
129  if (mp_mul (&t1, &x0, &t1) != MP_OKAY)

```

```

130     goto X1Y1;          /* t1 = (x1 + x0) * (y1 + y0) */
131
132     /* add x0y0 */
133     if (mp_add (&x0y0, &x1y1, &x0) != MP_OKAY)
134         goto X1Y1;      /* t2 = x0y0 + x1y1 */
135     if (s_mp_sub (&t1, &x0, &t1) != MP_OKAY)
136         goto X1Y1;      /* t1 = (x1+x0)*(y1+y0) - (x1y1 + x0y0) */
137
138     /* shift by B */
139     if (mp_lshd (&t1, B) != MP_OKAY)
140         goto X1Y1;      /* t1 = (x0y0 + x1y1 - (x1-x0)*(y1-y0))<<B */
141     if (mp_lshd (&x1y1, B * 2) != MP_OKAY)
142         goto X1Y1;      /* x1y1 = x1y1 << 2*B */
143
144     if (mp_add (&x0y0, &t1, &t1) != MP_OKAY)
145         goto X1Y1;      /* t1 = x0y0 + t1 */
146     if (mp_add (&t1, &x1y1, c) != MP_OKAY)
147         goto X1Y1;      /* t1 = x0y0 + t1 + x1y1 */
148
149     /* Algorithm succeeded set the return code to MP_OKAY */
150     err = MP_OKAY;
151
152     X1Y1:mp_clear (&x1y1);
153     X0Y0:mp_clear (&x0y0);
154     T1:mp_clear (&t1);
155     Y1:mp_clear (&y1);
156     Y0:mp_clear (&y0);
157     X1:mp_clear (&x1);
158     X0:mp_clear (&x0);
159     ERR:
160     return err;
161 }
162 #endif
163
164 /* ref:          tag: v1.0.1 */
165 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
166 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The new coding element in this routine, not seen in previous routines, is the usage of goto statements. The conventional wisdom is that goto statements should be avoided. This is generally true, however when every single function call can fail, it makes sense to handle error recovery with a single piece of

code. Lines 61 to 75 handle initializing all of the temporary variables required. Note how each of the if statements goes to a different label in case of failure. This allows the routine to correctly free only the temporaries that have been successfully allocated so far.

The temporary variables are all initialized using the `mp_init_size` routine since they are expected to be large. This saves the additional reallocation that would have been necessary. Also $x0$, $x1$, $y0$ and $y1$ have to be able to hold at least their respective number of digits for the next section of code.

The first algebraic portion of the algorithm is to split the two inputs into their halves. However, instead of using `mp_mod_2d` and `mp_rshd` to extract the halves, the respective code has been placed inline within the body of the function. To initialize the halves, the **used** and **sign** members are copied first. The first for loop on line 101 copies the lower halves. Since they are both the same magnitude it is simpler to calculate both lower halves in a single loop. The for loop on lines 106 and 106 calculate the upper halves $x1$ and $y1$ respectively.

By inlining the calculation of the halves, the Karatsuba multiplier has a slightly lower overhead and can be used for smaller magnitude inputs.

When line 150 is reached, the algorithm has completed successfully. The “error status” variable *err* is set to **MP_OKAY** so that the same code that handles errors can be used to clear the temporary variables and return.

5.2.5 Toom-Cook 3-Way Multiplication

Toom-Cook 3-Way [22] multiplication is essentially the polynomial basis algorithm for $n = 2$ except that the points are chosen such that ζ is easy to compute and the resulting system of equations easy to reduce. Here, the points ζ_0 , $16 \cdot \zeta_{\frac{1}{2}}$, ζ_1 , ζ_2 and ζ_∞ make up the five required points to solve for the coefficients of the $W(x)$.

With the five relations that Toom-Cook specifies, the following system of equations is formed.

$$\begin{array}{rclclclcl}
 \zeta_0 & = & 0w_4 & + & 0w_3 & + & 0w_2 & + & 0w_1 & + & 1w_0 \\
 16 \cdot \zeta_{\frac{1}{2}} & = & 1w_4 & + & 2w_3 & + & 4w_2 & + & 8w_1 & + & 16w_0 \\
 \zeta_1 & = & 1w_4 & + & 1w_3 & + & 1w_2 & + & 1w_1 & + & 1w_0 \\
 \zeta_2 & = & 16w_4 & + & 8w_3 & + & 4w_2 & + & 2w_1 & + & 1w_0 \\
 \zeta_\infty & = & 1w_4 & + & 0w_3 & + & 0w_2 & + & 0w_1 & + & 0w_0
 \end{array}$$

A trivial solution to this matrix requires 12 subtractions, two multiplications by a small power of two, two divisions by a small power of two, two divisions

by three and one multiplication by three. All of these 19 sub-operations require less than quadratic time, meaning that the algorithm can be faster than a baseline multiplication. However, the greater complexity of this algorithm places the cutoff point (**TOOM_MUL_CUTOFF**) where Toom-Cook becomes more efficient much higher than the Karatsuba cutoff point.

Algorithm **mp_toom_mul**.

Input. mp_int a and mp_int b

Output. $c \leftarrow a \cdot b$

Split a and b into three pieces. E.g. $a = a_2\beta^{2k} + a_1\beta^k + a_0$

1. $k \leftarrow \lfloor \min(a.used, b.used)/3 \rfloor$
2. $a_0 \leftarrow a \pmod{\beta^k}$
3. $a_1 \leftarrow \lfloor a/\beta^k \rfloor$, $a_1 \leftarrow a_1 \pmod{\beta^k}$
4. $a_2 \leftarrow \lfloor a/\beta^{2k} \rfloor$, $a_2 \leftarrow a_2 \pmod{\beta^k}$
5. $b_0 \leftarrow b \pmod{\beta^k}$
6. $b_1 \leftarrow \lfloor b/\beta^k \rfloor$, $b_1 \leftarrow b_1 \pmod{\beta^k}$
7. $b_2 \leftarrow \lfloor b/\beta^{2k} \rfloor$, $b_2 \leftarrow b_2 \pmod{\beta^k}$

Find the five equations for w_0, w_1, \dots, w_4 .

8. $w_0 \leftarrow a_0 \cdot b_0$
9. $w_4 \leftarrow a_2 \cdot b_2$
10. $tmp_1 \leftarrow 2 \cdot a_0$, $tmp_1 \leftarrow a_1 + tmp_1$, $tmp_1 \leftarrow 2 \cdot tmp_1$, $tmp_1 \leftarrow tmp_1 + a_2$
11. $tmp_2 \leftarrow 2 \cdot b_0$, $tmp_2 \leftarrow b_1 + tmp_2$, $tmp_2 \leftarrow 2 \cdot tmp_2$, $tmp_2 \leftarrow tmp_2 + b_2$
12. $w_1 \leftarrow tmp_1 \cdot tmp_2$
13. $tmp_1 \leftarrow 2 \cdot a_2$, $tmp_1 \leftarrow a_1 + tmp_1$, $tmp_1 \leftarrow 2 \cdot tmp_1$, $tmp_1 \leftarrow tmp_1 + a_0$
14. $tmp_2 \leftarrow 2 \cdot b_2$, $tmp_2 \leftarrow b_1 + tmp_2$, $tmp_2 \leftarrow 2 \cdot tmp_2$, $tmp_2 \leftarrow tmp_2 + b_0$
15. $w_3 \leftarrow tmp_1 \cdot tmp_2$
16. $tmp_1 \leftarrow a_0 + a_1$, $tmp_1 \leftarrow tmp_1 + a_2$, $tmp_2 \leftarrow b_0 + b_1$, $tmp_2 \leftarrow tmp_2 + b_2$
17. $w_2 \leftarrow tmp_1 \cdot tmp_2$

Continued on the next page.

Figure 5.8: Algorithm mp_toom_mul

Algorithm **mp_toom_mul** (continued).

Input. mp_int a and mp_int b

Output. $c \leftarrow a \cdot b$

Now solve the system of equations.

18. $w_1 \leftarrow w_4 - w_1, w_3 \leftarrow w_3 - w_0$
19. $w_1 \leftarrow \lfloor w_1/2 \rfloor, w_3 \leftarrow \lfloor w_3/2 \rfloor$
20. $w_2 \leftarrow w_2 - w_0, w_2 \leftarrow w_2 - w_4$
21. $w_1 \leftarrow w_1 - w_2, w_3 \leftarrow w_3 - w_2$
22. $tmp_1 \leftarrow 8 \cdot w_0, w_1 \leftarrow w_1 - tmp_1, tmp_1 \leftarrow 8 \cdot w_4, w_3 \leftarrow w_3 - tmp_1$
23. $w_2 \leftarrow 3 \cdot w_2, w_2 \leftarrow w_2 - w_1, w_2 \leftarrow w_2 - w_3$
24. $w_1 \leftarrow w_1 - w_2, w_3 \leftarrow w_3 - w_2$
25. $w_1 \leftarrow \lfloor w_1/3 \rfloor, w_3 \leftarrow \lfloor w_3/3 \rfloor$

Now substitute β^k for x by shifting w_0, w_1, \dots, w_4 .

26. for n from 1 to 4 do

26.1 $w_n \leftarrow w_n \cdot \beta^{nk}$

27. $c \leftarrow w_0 + w_1, c \leftarrow c + w_2, c \leftarrow c + w_3, c \leftarrow c + w_4$

28. Return(*MP_OKAY*)

Figure 5.9: Algorithm mp_toom_mul (continued)

Algorithm mp_toom_mul. This algorithm computes the product of two mp_int variables a and b using the Toom-Cook approach. Compared to the Karatsuba multiplication, this algorithm has a lower asymptotic running time of approximately $O(n^{1.464})$ but at an obvious cost in overhead. In this description, several statements have been compounded to save space. The intention is that the statements are executed from left to right across any given step.

The two inputs a and b are first split into three k -digit integers a_0, a_1, a_2 and b_0, b_1, b_2 respectively. From these smaller integers the coefficients of the polynomial basis representations $f(x)$ and $g(x)$ are known and can be used to find the relations required.

The first two relations w_0 and w_4 are the points ζ_0 and ζ_∞ respectively. The relation w_1, w_2 and w_3 correspond to the points $16 \cdot \zeta_{\frac{1}{2}}, \zeta_2$ and ζ_1 respectively. These are found using logical shifts to independently find $f(y)$ and $g(y)$ which significantly speeds up the algorithm.

After the five relations w_0, w_1, \dots, w_4 have been computed, the system they represent must be solved in order for the unknown coefficients w_1, w_2 and w_3 to be isolated. The steps 18 through 25 perform the system reduction required

as previously described. Each step of the reduction represents the comparable matrix operation that would be performed had this been performed by pencil. For example, step 18 indicates that row 1 must be subtracted from row 4 and simultaneously row 0 subtracted from row 3.

Once the coefficients have been isolated, the polynomial $W(x) = \sum_{i=0}^{2n} w_i x^i$ is known. By substituting β^k for x , the integer result $a \cdot b$ is produced.

File: bn_mp_toom_mul.c

```

016
017  /* multiplication using the Toom-Cook 3-way algorithm
018  *
019  * Much more complicated than Karatsuba but has a lower
020  * asymptotic running time of  $O(N^{1.464})$ . This algorithm is
021  * only particularly useful on VERY large inputs
022  * (we're talking 1000s of digits here...).
023  */
024  int mp_toom_mul(mp_int *a, mp_int *b, mp_int *c)
025  {
026      mp_int w0, w1, w2, w3, w4, tmp1, tmp2, a0, a1, a2, b0, b1, b2;
027      int res, B;
028
029      /* init temps */
030      if ((res = mp_init_multi(&w0, &w1, &w2, &w3, &w4,
031                              &a0, &a1, &a2, &b0, &b1,
032                              &b2, &tmp1, &tmp2, NULL)) != MP_OKAY) {
033          return res;
034      }
035
036      /* B */
037      B = MIN(a->used, b->used) / 3;
038
039      /* a = a2 * B**2 + a1 * B + a0 */
040      if ((res = mp_mod_2d(a, DIGIT_BIT * B, &a0)) != MP_OKAY) {
041          goto ERR;
042      }
043
044      if ((res = mp_copy(a, &a1)) != MP_OKAY) {
045          goto ERR;
046      }
047      mp_rshd(&a1, B);
048      if ((res = mp_mod_2d(&a1, DIGIT_BIT * B, &a1)) != MP_OKAY) {
049          goto ERR;

```

```

050     }
051
052     if ((res = mp_copy(a, &a2)) != MP_OKAY) {
053         goto ERR;
054     }
055     mp_rshd(&a2, B*2);
056
057     /* b = b2 * B**2 + b1 * B + b0 */
058     if ((res = mp_mod_2d(b, DIGIT_BIT * B, &b0)) != MP_OKAY) {
059         goto ERR;
060     }
061
062     if ((res = mp_copy(b, &b1)) != MP_OKAY) {
063         goto ERR;
064     }
065     mp_rshd(&b1, B);
066     (void)mp_mod_2d(&b1, DIGIT_BIT * B, &b1);
067
068     if ((res = mp_copy(b, &b2)) != MP_OKAY) {
069         goto ERR;
070     }
071     mp_rshd(&b2, B*2);
072
073     /* w0 = a0*b0 */
074     if ((res = mp_mul(&a0, &b0, &w0)) != MP_OKAY) {
075         goto ERR;
076     }
077
078     /* w4 = a2 * b2 */
079     if ((res = mp_mul(&a2, &b2, &w4)) != MP_OKAY) {
080         goto ERR;
081     }
082
083     /* w1 = (a2 + 2(a1 + 2a0))(b2 + 2(b1 + 2b0)) */
084     if ((res = mp_mul_2(&a0, &tmp1)) != MP_OKAY) {
085         goto ERR;
086     }
087     if ((res = mp_add(&tmp1, &a1, &tmp1)) != MP_OKAY) {
088         goto ERR;
089     }
090     if ((res = mp_mul_2(&tmp1, &tmp1)) != MP_OKAY) {
091         goto ERR;

```

```

092     }
093     if ((res = mp_add(&tmp1, &a2, &tmp1)) != MP_OKAY) {
094         goto ERR;
095     }
096
097     if ((res = mp_mul_2(&b0, &tmp2)) != MP_OKAY) {
098         goto ERR;
099     }
100     if ((res = mp_add(&tmp2, &b1, &tmp2)) != MP_OKAY) {
101         goto ERR;
102     }
103     if ((res = mp_mul_2(&tmp2, &tmp2)) != MP_OKAY) {
104         goto ERR;
105     }
106     if ((res = mp_add(&tmp2, &b2, &tmp2)) != MP_OKAY) {
107         goto ERR;
108     }
109
110     if ((res = mp_mul(&tmp1, &tmp2, &w1)) != MP_OKAY) {
111         goto ERR;
112     }
113
114     /* w3 = (a0 + 2(a1 + 2a2))(b0 + 2(b1 + 2b2)) */
115     if ((res = mp_mul_2(&a2, &tmp1)) != MP_OKAY) {
116         goto ERR;
117     }
118     if ((res = mp_add(&tmp1, &a1, &tmp1)) != MP_OKAY) {
119         goto ERR;
120     }
121     if ((res = mp_mul_2(&tmp1, &tmp1)) != MP_OKAY) {
122         goto ERR;
123     }
124     if ((res = mp_add(&tmp1, &a0, &tmp1)) != MP_OKAY) {
125         goto ERR;
126     }
127
128     if ((res = mp_mul_2(&b2, &tmp2)) != MP_OKAY) {
129         goto ERR;
130     }
131     if ((res = mp_add(&tmp2, &b1, &tmp2)) != MP_OKAY) {
132         goto ERR;
133     }

```

```

134     if ((res = mp_mul_2(&tmp2, &tmp2)) != MP_OKAY) {
135         goto ERR;
136     }
137     if ((res = mp_add(&tmp2, &b0, &tmp2)) != MP_OKAY) {
138         goto ERR;
139     }
140
141     if ((res = mp_mul(&tmp1, &tmp2, &w3)) != MP_OKAY) {
142         goto ERR;
143     }
144
145
146     /* w2 = (a2 + a1 + a0)(b2 + b1 + b0) */
147     if ((res = mp_add(&a2, &a1, &tmp1)) != MP_OKAY) {
148         goto ERR;
149     }
150     if ((res = mp_add(&tmp1, &a0, &tmp1)) != MP_OKAY) {
151         goto ERR;
152     }
153     if ((res = mp_add(&b2, &b1, &tmp2)) != MP_OKAY) {
154         goto ERR;
155     }
156     if ((res = mp_add(&tmp2, &b0, &tmp2)) != MP_OKAY) {
157         goto ERR;
158     }
159     if ((res = mp_mul(&tmp1, &tmp2, &w2)) != MP_OKAY) {
160         goto ERR;
161     }
162
163     /* now solve the matrix
164
165         0  0  0  0  1
166         1  2  4  8 16
167         1  1  1  1  1
168        16  8  4  2  1
169         1  0  0  0  0
170
171         using 12 subtractions, 4 shifts,
172             2 small divisions and 1 small multiplication
173     */
174
175     /* r1 - r4 */

```

```

176     if ((res = mp_sub(&w1, &w4, &w1)) != MP_OKAY) {
177         goto ERR;
178     }
179     /* r3 - r0 */
180     if ((res = mp_sub(&w3, &w0, &w3)) != MP_OKAY) {
181         goto ERR;
182     }
183     /* r1/2 */
184     if ((res = mp_div_2(&w1, &w1)) != MP_OKAY) {
185         goto ERR;
186     }
187     /* r3/2 */
188     if ((res = mp_div_2(&w3, &w3)) != MP_OKAY) {
189         goto ERR;
190     }
191     /* r2 - r0 - r4 */
192     if ((res = mp_sub(&w2, &w0, &w2)) != MP_OKAY) {
193         goto ERR;
194     }
195     if ((res = mp_sub(&w2, &w4, &w2)) != MP_OKAY) {
196         goto ERR;
197     }
198     /* r1 - r2 */
199     if ((res = mp_sub(&w1, &w2, &w1)) != MP_OKAY) {
200         goto ERR;
201     }
202     /* r3 - r2 */
203     if ((res = mp_sub(&w3, &w2, &w3)) != MP_OKAY) {
204         goto ERR;
205     }
206     /* r1 - 8r0 */
207     if ((res = mp_mul_2d(&w0, 3, &tmp1)) != MP_OKAY) {
208         goto ERR;
209     }
210     if ((res = mp_sub(&w1, &tmp1, &w1)) != MP_OKAY) {
211         goto ERR;
212     }
213     /* r3 - 8r4 */
214     if ((res = mp_mul_2d(&w4, 3, &tmp1)) != MP_OKAY) {
215         goto ERR;
216     }
217     if ((res = mp_sub(&w3, &tmp1, &w3)) != MP_OKAY) {

```

```

218         goto ERR;
219     }
220     /* 3r2 - r1 - r3 */
221     if ((res = mp_mul_d(&w2, 3, &w2)) != MP_OKAY) {
222         goto ERR;
223     }
224     if ((res = mp_sub(&w2, &w1, &w2)) != MP_OKAY) {
225         goto ERR;
226     }
227     if ((res = mp_sub(&w2, &w3, &w2)) != MP_OKAY) {
228         goto ERR;
229     }
230     /* r1 - r2 */
231     if ((res = mp_sub(&w1, &w2, &w1)) != MP_OKAY) {
232         goto ERR;
233     }
234     /* r3 - r2 */
235     if ((res = mp_sub(&w3, &w2, &w3)) != MP_OKAY) {
236         goto ERR;
237     }
238     /* r1/3 */
239     if ((res = mp_div_3(&w1, &w1, NULL)) != MP_OKAY) {
240         goto ERR;
241     }
242     /* r3/3 */
243     if ((res = mp_div_3(&w3, &w3, NULL)) != MP_OKAY) {
244         goto ERR;
245     }
246
247     /* at this point shift W[n] by B*n */
248     if ((res = mp_lshd(&w1, 1*B)) != MP_OKAY) {
249         goto ERR;
250     }
251     if ((res = mp_lshd(&w2, 2*B)) != MP_OKAY) {
252         goto ERR;
253     }
254     if ((res = mp_lshd(&w3, 3*B)) != MP_OKAY) {
255         goto ERR;
256     }
257     if ((res = mp_lshd(&w4, 4*B)) != MP_OKAY) {
258         goto ERR;
259     }

```

```

260
261     if ((res = mp_add(&w0, &w1, c)) != MP_OKAY) {
262         goto ERR;
263     }
264     if ((res = mp_add(&w2, &w3, &tmp1)) != MP_OKAY) {
265         goto ERR;
266     }
267     if ((res = mp_add(&w4, &tmp1, &tmp1)) != MP_OKAY) {
268         goto ERR;
269     }
270     if ((res = mp_add(&tmp1, c, c)) != MP_OKAY) {
271         goto ERR;
272     }
273
274 ERR:
275     mp_clear_multi(&w0, &w1, &w2, &w3, &w4,
276                  &a0, &a1, &a2, &b0, &b1,
277                  &b2, &tmp1, &tmp2, NULL);
278     return res;
279 }
280
281 #endif
282
283 /* ref:          tag: v1.0.1 */
284 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
285 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The first obvious thing to note is that this algorithm is complicated. The complexity is worth it if you are multiplying very large numbers. For example, a 10,000 digit multiplication takes approximately 99,282,205 fewer single precision multiplications with Toom–Cook than a Comba or baseline approach (this is a savings of more than 99%). For most “crypto” sized numbers this algorithm is not practical as Karatsuba has a much lower cutoff point.

First we split a and b into three roughly equal portions. This has been accomplished (lines 40 to 71) with combinations of `mp_rshd()` and `mp_mod_2d()` function calls. At this point $a = a_2 \cdot \beta^2 + a_1 \cdot \beta + a_0$ and similarly for b .

Next we compute the five points w_0, w_1, w_2, w_3 and w_4 . Recall that w_0 and w_4 can be computed directly from the portions so we get those out of the way first (lines 74 and 79). Next we compute w_1, w_2 and w_3 using Horner’s method.

After this point we solve for the actual values of w_1, w_2 and w_3 by reducing the 5×5 system which is relatively straight forward.

5.2.6 Signed Multiplication

Now that algorithms to handle multiplications of every useful dimensions have been developed, a rather simple finishing touch is required. So far all of the multiplication algorithms have been unsigned multiplications which leaves only a signed multiplication algorithm to be established.

Algorithm **mp_mul**.

Input. mp_int a and mp_int b

Output. $c \leftarrow a \cdot b$

1. If $a.sign = b.sign$ then
 - 1.1 $sign = MP_ZPOS$
 2. else
 - 2.1 $sign = MP_ZNEG$
 3. If $\min(a.used, b.used) \geq TOOM_MUL_CUTOFF$ then
 - 3.1 $c \leftarrow a \cdot b$ using algorithm mp_toom_mul
 4. else if $\min(a.used, b.used) \geq KARATSUBA_MUL_CUTOFF$ then
 - 4.1 $c \leftarrow a \cdot b$ using algorithm mp_karatsuba_mul
 5. else
 - 5.1 $digs \leftarrow a.used + b.used + 1$
 - 5.2 If $digs < MP_ARRAY$ and $\min(a.used, b.used) \leq \delta$ then
 - 5.2.1 $c \leftarrow a \cdot b \pmod{\beta^{digs}}$ using algorithm fast_s_mp_mul_digs.
 - 5.3 else
 - 5.3.1 $c \leftarrow a \cdot b \pmod{\beta^{digs}}$ using algorithm s_mp_mul_digs.
 6. $c.sign \leftarrow sign$
 7. Return the result of the unsigned multiplication performed.
-

Figure 5.10: Algorithm mp_mul

Algorithm mp_mul. This algorithm performs the signed multiplication of two inputs. It will make use of any of the three unsigned multiplication algorithms available when the input is of appropriate size. The **sign** of the result is not set until the end of the algorithm since algorithm s_mp_mul_digs will clear it.

File: bn_mp_mul.c

```

016
017  /* high level multiplication (handles sign) */
018  int mp_mul (mp_int * a, mp_int * b, mp_int * c)
019  {

```



```

020     int      res, neg;
021     neg = (a->sign == b->sign) ? MP_ZPOS : MP_NEG;
022
023     /* use Toom-Cook? */
024     #ifdef BN_MP_TOOM_MUL_C
025     if (MIN (a->used, b->used) >= TOOM_MUL_CUTOFF) {
026         res = mp_toom_mul(a, b, c);
027     } else
028     #endif
029     #ifdef BN_MP_KARATSUBA_MUL_C
030     /* use Karatsuba? */
031     if (MIN (a->used, b->used) >= KARATSUBA_MUL_CUTOFF) {
032         res = mp_karatsuba_mul (a, b, c);
033     } else
034     #endif
035     {
036         /* can we use the fast multiplier?
037          *
038          * The fast multiplier can be used if the output will
039          * have less than MP_WARRAY digits and the number of
040          * digits won't affect carry propagation
041          */
042         int      digs = a->used + b->used + 1;
043
044         #ifdef BN_FAST_S_MP_MUL_DIGS_C
045         if ((digs < MP_WARRAY) &&
046             (MIN(a->used, b->used) <=
047              (1 << ((CHAR_BIT * sizeof(mp_word)) - (2 * DIGIT_BIT))))) {
048             res = fast_s_mp_mul_digs (a, b, c, digs);
049         } else
050         #endif
051         {
052             #ifdef BN_S_MP_MUL_DIGS_C
053             res = s_mp_mul (a, b, c); /* uses s_mp_mul_digs */
054             #else
055             res = MP_VAL;
056             #endif
057         }
058     }
059     c->sign = (c->used > 0) ? neg : MP_ZPOS;
060     return res;
061 }

```

```

062  #endif
063
064  /* ref:          tag: v1.0.1 */
065  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
066  /* commit time: 2017-08-29 22:27:36 +0200 */

```

The implementation is rather simplistic and is not particularly noteworthy. Line 23 computes the sign of the result using the “?” operator from the C programming language. Line 47 computes δ using the fact that $1 \ll k$ is equal to 2^k .

5.3 Squaring

Squaring is a special case of multiplication where both multiplicands are equal. At first it may seem like there is no significant optimization available but in fact there is. Consider the multiplication of 576 against 241. In total there will be nine single precision multiplications performed which are $1 \cdot 6$, $1 \cdot 7$, $1 \cdot 5$, $4 \cdot 6$, $4 \cdot 7$, $4 \cdot 5$, $2 \cdot 6$, $2 \cdot 7$ and $2 \cdot 5$. Now consider the multiplication of 123 against 123. The nine products are $3 \cdot 3$, $3 \cdot 2$, $3 \cdot 1$, $2 \cdot 3$, $2 \cdot 2$, $2 \cdot 1$, $1 \cdot 3$, $1 \cdot 2$ and $1 \cdot 1$. On closer inspection some of the products are equivalent. For example, $3 \cdot 2 = 2 \cdot 3$ and $3 \cdot 1 = 1 \cdot 3$.

For any n -digit input, there are $\frac{(n^2+n)}{2}$ possible unique single precision multiplications required compared to the n^2 required for multiplication. The following diagram gives an example of the operations required.

| | | 1 | 2 | 3 | |
|-------------|-------------|-------------|-------------|-------------|-------|
| \times | | 1 | 2 | 3 | |
| | | $3 \cdot 1$ | $3 \cdot 2$ | $3 \cdot 3$ | Row 0 |
| | $2 \cdot 1$ | $2 \cdot 2$ | $2 \cdot 3$ | | Row 1 |
| $1 \cdot 1$ | $1 \cdot 2$ | $1 \cdot 3$ | | | Row 2 |

Figure 5.11: Squaring Optimization Diagram

Starting from zero and numbering the columns from right to left a very simple pattern becomes obvious. For the purposes of this discussion let x represent the number being squared. The first observation is that in row k the $2k$ 'th column of the product has a $(x_k)^2$ term in it.

The second observation is that every column j in row k where $j \neq 2k$ is part of a double product. Every non-square term of a column will appear twice

hence the name “double product”. Every odd column is made up entirely of double products. In fact every column is made up of double products and at most one square (*see the exercise section*).

The third and final observation is that for row k the first unique non-square term, that is, one that hasn’t already appeared in an earlier row, occurs at column $2k + 1$. For example, on row 1 of the previous squaring, column one is part of the double product with column one from row zero. Column two of row one is a square and column three is the first unique column.

5.3.1 The Baseline Squaring Algorithm

The baseline squaring algorithm is meant to be a catch-all squaring algorithm. It will handle any of the input sizes that the faster routines will not handle.

Algorithm s_mp_sqr. This algorithm computes the square of an input using the three observations on squaring. It is based fairly faithfully on algorithm 14.16 of HAC [2, pp.596-597]. Similar to algorithm s_mp_mul_digs, a temporary mp_int is allocated to hold the result of the squaring. This allows the destination mp_int to be the same as the source mp_int.

The outer loop of this algorithm begins on step 4. It is best to think of the outer loop as walking down the rows of the partial results, while the inner loop computes the columns of the partial result. Step 4.1 and 4.2 compute the square term for each row, and step 4.3 and 4.4 propagate the carry and compute the double products.

The requirement that a mp_word be able to represent the range $0 \leq x < 2\beta^2$ arises from this very algorithm. The product $a_{ix}a_{iy}$ will lie in the range $0 \leq x \leq \beta^2 - 2\beta + 1$ which is obviously less than β^2 meaning that when it is multiplied by two, it can be properly represented by a mp_word.

Similar to algorithm s_mp_mul_digs, after every pass of the inner loop, the destination is correctly set to the sum of all of the partial results calculated so far. This involves expensive carry propagation which will be eliminated in the next algorithm.

File: bn_s_mp_sqr.c

```

016
017  /* low level squaring, b = a*a, HAC pp.596-597, Algorithm 14.16 */
018  int s_mp_sqr (mp_int * a, mp_int * b)
019  {
020      mp_int  t;
021      int     res, ix, iy, pa;
022      mp_word r;

```

Algorithm **s_mp_sqr**.

Input. mp_int a

Output. $b \leftarrow a^2$

1. Init a temporary mp_int of at least $2 \cdot a.used + 1$ digits. (*mp_init_size*)
 2. If step 1 failed return(*MP_MEM*)
 3. $t.used \leftarrow 2 \cdot a.used + 1$
 4. For ix from 0 to $a.used - 1$ do
 - Calculate the square.
 - 4.1 $\hat{r} \leftarrow t_{2ix} + (a_{ix})^2$
 - 4.2 $t_{2ix} \leftarrow \hat{r} \pmod{\beta}$
 - Calculate the double products after the square.
 - 4.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
 - 4.4 For iy from $ix + 1$ to $a.used - 1$ do
 - 4.4.1 $\hat{r} \leftarrow 2 \cdot a_{ix}a_{iy} + t_{ix+iy} + u$
 - 4.4.2 $t_{ix+iy} \leftarrow \hat{r} \pmod{\beta}$
 - 4.4.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
 - Set the last carry.
 - 4.5 While $u > 0$ do
 - 4.5.1 $iy \leftarrow iy + 1$
 - 4.5.2 $\hat{r} \leftarrow t_{ix+iy} + u$
 - 4.5.3 $t_{ix+iy} \leftarrow \hat{r} \pmod{\beta}$
 - 4.5.4 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
 5. Clamp excess digits of t . (*mp_clamp*)
 6. Exchange b and t .
 7. Clear t (*mp_clear*)
 8. Return(*MP_OKAY*)
-

Figure 5.12: Algorithm s_mp_sqr

```

023     mp_digit u, tmpx, *tmpt;
024
025     pa = a->used;
026     if ((res = mp_init_size (&t, (2 * pa) + 1)) != MP_OKAY) {
027         return res;
028     }
029
030     /* default used is maximum possible size */
031     t.used = (2 * pa) + 1;
032
```

```

033     for (ix = 0; ix < pa; ix++) {
034         /* first calculate the digit at 2*ix */
035         /* calculate double precision result */
036         r = (mp_word)t.dp[2*ix] +
037             ((mp_word)a->dp[ix] * (mp_word)a->dp[ix]);
038
039         /* store lower part in result */
040         t.dp[ix+ix] = (mp_digit) (r & ((mp_word) MP_MASK));
041
042         /* get the carry */
043         u      = (mp_digit)(r >> ((mp_word) DIGIT_BIT));
044
045         /* left hand side of A[ix] * A[iy] */
046         tmpx    = a->dp[ix];
047
048         /* alias for where to store the results */
049         tmpt    = t.dp + ((2 * ix) + 1);
050
051         for (iy = ix + 1; iy < pa; iy++) {
052             /* first calculate the product */
053             r      = ((mp_word)tmpx) * ((mp_word)a->dp[iy]);
054
055             /* now calculate the double precision result, note we use
056              * addition instead of *2 since it's easier to optimize
057              */
058             r      = ((mp_word) *tmpt) + r + r + ((mp_word) u);
059
060             /* store lower part */
061             *tmpt++ = (mp_digit) (r & ((mp_word) MP_MASK));
062
063             /* get carry */
064             u      = (mp_digit)(r >> ((mp_word) DIGIT_BIT));
065         }
066         /* propagate upwards */
067         while (u != ((mp_digit) 0)) {
068             r      = ((mp_word) *tmpt) + ((mp_word) u);
069             *tmpt++ = (mp_digit) (r & ((mp_word) MP_MASK));
070             u      = (mp_digit)(r >> ((mp_word) DIGIT_BIT));
071         }
072     }
073
074     mp_clamp (&t);

```

```

075     mp_exch (&t, b);
076     mp_clear (&t);
077     return MP_OKAY;
078 }
079 #endif
080
081 /* ref:          tag: v1.0.1 */
082 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
083 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Inside the outer loop (line 33) the square term is calculated on line 36. The carry (line 43) has been extracted from the `mp_word` accumulator using a right shift. Aliases for a_{ix} and t_{ix+iy} are initialized (lines 46 and 49) to simplify the inner loop. The doubling is performed using two additions (line 58) since it is usually faster than shifting, if not at least as fast.

The important observation is that the inner loop does not begin at $iy = 0$ like for multiplication. As such the inner loops get progressively shorter as the algorithm proceeds. This is what leads to the savings compared to using a multiplication to square a number.

5.3.2 Faster Squaring by the “Comba” Method

A major drawback to the baseline method is the requirement for single precision shifting inside the $O(n^2)$ nested loop. Squaring has an additional drawback that it must double the product inside the inner loop as well. As for multiplication, the Comba technique can be used to eliminate these performance hazards.

The first obvious solution is to make an array of `mp_words` which will hold all of the columns. This will indeed eliminate all of the carry propagation operations from the inner loop. However, the inner product must still be doubled $O(n^2)$ times. The solution stems from the simple fact that $2a + 2b + 2c = 2(a + b + c)$. That is the sum of all of the double products is equal to double the sum of all the products. For example, $ab + ba + ac + ca = 2ab + 2ac = 2(ab + ac)$.

However, we cannot simply double all of the columns, since the squares appear only once per row. The most practical solution is to have two `mp_word` arrays. One array will hold the squares and the other array will hold the double products. With both arrays the doubling and carry propagation can be moved to a $O(n)$ work level outside the $O(n^2)$ level. In this case, we have an even simpler solution in mind.

Algorithm **fast_s_mp_sqr**.

Input. mp_int a

Output. $b \leftarrow a^2$

Place an array of **MP_WARRAY** mp.digits named W on the stack.

1. If $b.alloc < 2a.used + 1$ then grow b to $2a.used + 1$ digits. (*mp_grow*).
 2. If step 1 failed return(*MP_MEM*).
 3. $pa \leftarrow 2 \cdot a.used$
 4. $\hat{W}1 \leftarrow 0$
 5. for ix from 0 to $pa - 1$ do
 - 5.1 $\hat{W} \leftarrow 0$
 - 5.2 $ty \leftarrow \text{MIN}(a.used - 1, ix)$
 - 5.3 $tx \leftarrow ix - ty$
 - 5.4 $iy \leftarrow \text{MIN}(a.used - tx, ty + 1)$
 - 5.5 $iy \leftarrow \text{MIN}(iy, \lfloor (ty - tx + 1) / 2 \rfloor)$
 - 5.6 for iz from 0 to $iz - 1$ do
 - 5.6.1 $\hat{W} \leftarrow \hat{W} + a_{tx+iz}a_{ty-iz}$
 - 5.7 $\hat{W} \leftarrow 2 \cdot \hat{W} + \hat{W}1$
 - 5.8 if ix is even then
 - 5.8.1 $\hat{W} \leftarrow \hat{W} + (a_{\lfloor ix/2 \rfloor})^2$
 - 5.9 $W_{ix} \leftarrow \hat{W} \pmod{\beta}$
 - 5.10 $\hat{W}1 \leftarrow \lfloor \hat{W} / \beta \rfloor$
 6. $oldused \leftarrow b.used$
 7. $b.used \leftarrow 2 \cdot a.used$
 8. for ix from 0 to $pa - 1$ do
 - 8.1 $b_{ix} \leftarrow W_{ix}$
 9. for ix from pa to $oldused - 1$ do
 - 9.1 $b_{ix} \leftarrow 0$
 10. Clamp excess digits from b . (*mp_clamp*)
 11. Return(*MP_OKAY*).
-

Figure 5.13: Algorithm fast_s_mp_sqr

Algorithm fast_s_mp_sqr. This algorithm computes the square of an input using the Comba technique. It is designed to be a replacement for algorithm s_mp_sqr when the number of input digits is less than **MP_WARRAY** and less than $\frac{\delta}{2}$. This algorithm is very similar to the Comba multiplier except with a few key differences we shall make note of.

First, we have an accumulator and carry variables \hat{W} and $\hat{W}1$ respectively. This is because the inner loop products are to be doubled. If we had added the previous carry in we would be doubling too much. Next we perform an addition MIN condition on iy (step 5.5) to prevent overlapping digits. For example, $a_3 \cdot a_5$ is equal $a_5 \cdot a_3$. Whereas in the multiplication case we would have $5 < a_{\text{used}}$ and $3 \geq 0$ is maintained since we double the sum of the products just outside the inner loop we have to avoid doing this. This is also a good thing since we perform fewer multiplications and the routine ends up being faster.

Finally the last difference is the addition of the “square” term outside the inner loop (step 5.8). We add in the square only to even outputs and it is the square of the term at the $\lfloor ix/2 \rfloor$ position.

File: bn_fast_s_mp_sqr.c

```

016
017 /* the jist of squaring...
018  * you do like mult except the offset of the tmpx [one that
019  * starts closer to zero] can't equal the offset of tmpy.
020  * So basically you set up iy like before then you min it with
021  * (ty-tx) so that it never happens. You double all those
022  * you add in the inner loop
023
024 After that loop you do the squares and add them in.
025 */
026
027 int fast_s_mp_sqr (mp_int * a, mp_int * b)
028 {
029     int      olduse, res, pa, ix, iz;
030     mp_digit  W[MP_WARRAY], *tmpx;
031     mp_word   W1;
032
033     /* grow the destination as required */
034     pa = a->used + a->used;
035     if (b->alloc < pa) {
036         if ((res = mp_grow (b, pa)) != MP_OKAY) {
037             return res;
038         }
039     }
040
041     /* number of output digits to produce */
042     W1 = 0;
043     for (ix = 0; ix < pa; ix++) {
044         int      tx, ty, iy;
```



```

045     mp_word  _W;
046     mp_digit *tmpy;
047
048     /* clear counter */
049     _W = 0;
050
051     /* get offsets into the two bignums */
052     ty = MIN(a->used-1, ix);
053     tx = ix - ty;
054
055     /* setup temp aliases */
056     tmpx = a->dp + tx;
057     tmpy = a->dp + ty;
058
059     /* this is the number of times the loop will iterate, essentially
060        while (tx++ < a->used && ty-- >= 0) { ... }
061     */
062     iy = MIN(a->used-tx, ty+1);
063
064     /* now for squaring tx can never equal ty
065        * we halve the distance since they approach at a rate of 2x
066        * and we have to round because odd cases need to be executed
067     */
068     iy = MIN(iy, ((ty-tx)+1)>>1);
069
070     /* execute loop */
071     for (iz = 0; iz < iy; iz++) {
072         _W += ((mp_word)*tmpx++)*((mp_word)*tmpy--);
073     }
074
075     /* double the inner product and add carry */
076     _W = _W + _W + W1;
077
078     /* even columns have the square term in them */
079     if ((ix&1) == 0) {
080         _W += ((mp_word)a->dp[ix>>1])*((mp_word)a->dp[ix>>1]);
081     }
082
083     /* store it */
084     W[ix] = (mp_digit)(_W & MP_MASK);
085
086     /* make next carry */

```

```

087         W1 = _W >> ((mp_word)DIGIT_BIT);
088     }
089
090     /* setup dest */
091     olduse = b->used;
092     b->used = a->used+a->used;
093
094     {
095         mp_digit *tmpb;
096         tmpb = b->dp;
097         for (ix = 0; ix < pa; ix++) {
098             *tmpb++ = W[ix] & MP_MASK;
099         }
100
101         /* clear unused digits [that existed in the old copy of c] */
102         for (; ix < olduse; ix++) {
103             *tmpb++ = 0;
104         }
105     }
106     mp_clamp(b);
107     return MP_OKAY;
108 }
109 #endif
110
111 /* ref:          tag: v1.0.1 */
112 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
113 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This implementation is essentially a copy of Comba multiplication with the appropriate changes added to make it faster for the special case of squaring.

5.3.3 Polynomial Basis Squaring

The same algorithm that performs optimal polynomial basis multiplication can be used to perform polynomial basis squaring. The minor exception is that $\zeta_y = f(y)g(y)$ is actually equivalent to $\zeta_y = f(y)^2$ since $f(y) = g(y)$. Instead of performing $2n + 1$ multiplications to find the ζ relations, squaring operations are performed instead.

5.3.4 Karatsuba Squaring

Let $f(x) = ax + b$ represent the polynomial basis representation of a number to square. Let $h(x) = (f(x))^2$ represent the square of the polynomial. The Karatsuba equation can be modified to square a number with the following equation.

$$h(x) = a^2x^2 + ((a+b)^2 - (a^2 + b^2))x + b^2 \quad (5.7)$$

Upon closer inspection this equation only requires the calculation of three half-sized squares: a^2 , b^2 and $(a+b)^2$. As in Karatsuba multiplication, this algorithm can be applied recursively on the input and will achieve an asymptotic running time of $O(n^{\lg(3)})$.

If the asymptotic times of Karatsuba squaring and multiplication are the same, why not simply use the multiplication algorithm instead? The answer to this arises from the cutoff point for squaring. As in multiplication there exists a cutoff point, at which the time required for a Comba based squaring and a Karatsuba based squaring meet. Due to the overhead inherent in the Karatsuba method, the cutoff point is fairly high. For example, on an AMD Athlon XP processor with $\beta = 2^{28}$, the cutoff point is around 127 digits.

Consider squaring a 200 digit number with this technique. It will be split into two 100 digit halves which are subsequently squared. The 100 digit halves will not be squared using Karatsuba, but instead using the faster Comba based squaring algorithm. If Karatsuba multiplication were used instead, the 100 digit numbers would be squared with a slower Comba based multiplication.

Algorithm **mp_karatsuba_sqr**.

Input. mp_int a

Output. $b \leftarrow a^2$

1. Initialize the following temporary mp_ints: $x0$, $x1$, $t1$, $t2$, $x0x0$ and $x1x1$.
2. If any of the initializations on step 1 failed return(MP_MEM).

Split the input. e.g. $a = x1\beta^B + x0$

3. $B \leftarrow \lfloor a.used/2 \rfloor$
4. $x0 \leftarrow a \pmod{\beta^B}$ (mp_mod_2d)
5. $x1 \leftarrow \lfloor a/\beta^B \rfloor$ (mp_lshd)

Calculate the three squares.

6. $x0x0 \leftarrow x0^2$ (mp_sqr)
7. $x1x1 \leftarrow x1^2$
8. $t1 \leftarrow x1 + x0$ (s_mp_add)
9. $t1 \leftarrow t1^2$

Compute the middle term.

10. $t2 \leftarrow x0x0 + x1x1$ (s_mp_add)
11. $t1 \leftarrow t1 - t2$

Compute final product.

12. $t1 \leftarrow t1\beta^B$ (mp_lshd)
 13. $x1x1 \leftarrow x1x1\beta^{2B}$
 14. $t1 \leftarrow t1 + x0x0$
 15. $b \leftarrow t1 + x1x1$
 16. Return(MP_OKAY).
-

Figure 5.14: Algorithm mp_karatsuba_sqr

Algorithm mp_karatsuba_sqr. This algorithm computes the square of an input a using the Karatsuba technique. This algorithm is very similar to the Karatsuba based multiplication algorithm with the exception that the three half-size multiplications have been replaced with three half-size squarings.

The radix point for squaring is simply placed exactly in the middle of the digits when the input has an odd number of digits, otherwise it is placed just below the middle. Step 3, 4 and 5 compute the two halves required using B as the radix point. The first two squares in steps 6 and 7 are rather straightforward while the last square is of a more compact form.

By expanding $(x1 + x0)^2$, the $x1^2$ and $x0^2$ terms in the middle disappear, that is $(x0 - x1)^2 - (x1^2 + x0^2) = 2 \cdot x0 \cdot x1$. Now if $5n$ single precision additions and a squaring of n -digits is faster than multiplying two n -digit numbers and doubling then this method is faster. Assuming no further recursions occur, the difference can be estimated with the following inequality.

Let p represent the cost of a single precision addition and q the cost of a single precision multiplication both in terms of time³.

$$5pn + \frac{q(n^2 + n)}{2} \leq pn + qn^2 \quad (5.8)$$

For example, on an AMD Athlon XP processor $p = \frac{1}{3}$ and $q = 6$. This implies that the following inequality should hold.

$$\begin{array}{rcl} \frac{5n}{3} + 3n^2 + 3n & < & \frac{n}{3} + 6n^2 \\ \frac{5}{3} + 3n + 3 & < & \frac{1}{3} + 6n \\ \frac{13}{9} & < & n \end{array}$$

This results in a cutoff point around $n = 2$. As a consequence it is actually faster to compute the middle term the “long way” on processors where multiplication is substantially slower⁴ than simpler operations such as addition.

File: bn_mp_karatsuba_sqr.c

```

016
017  /* Karatsuba squaring, computes b = a*a using three
018   * half size squarings
019   *
020   * See comments of karatsuba_mul for details. It
021   * is essentially the same algorithm but merely
022   * tuned to perform recursive squarings.
023   */
024  int mp_karatsuba_sqr (mp_int * a, mp_int * b)
025  {
026      mp_int  x0, x1, t1, t2, x0x0, x1x1;
027      int      B, err;
028
029      err = MP_MEM;
030
```

³Or machine clock cycles.

⁴On the Athlon there is a 1:17 ratio between clock cycles for addition and multiplication. On the Intel P4 processor this ratio is 1:29 making this method even more beneficial. The only common exception is the ARMv4 processor which has a ratio of 1:7.

```

031  /* min # of digits */
032  B = a->used;
033
034  /* now divide in two */
035  B = B >> 1;
036
037  /* init copy all the temps */
038  if (mp_init_size (&x0, B) != MP_OKAY)
039      goto ERR;
040  if (mp_init_size (&x1, a->used - B) != MP_OKAY)
041      goto X0;
042
043  /* init temps */
044  if (mp_init_size (&t1, a->used * 2) != MP_OKAY)
045      goto X1;
046  if (mp_init_size (&t2, a->used * 2) != MP_OKAY)
047      goto T1;
048  if (mp_init_size (&x0x0, B * 2) != MP_OKAY)
049      goto T2;
050  if (mp_init_size (&x1x1, (a->used - B) * 2) != MP_OKAY)
051      goto X0X0;
052
053  {
054      int x;
055      mp_digit *dst, *src;
056
057      src = a->dp;
058
059      /* now shift the digits */
060      dst = x0.dp;
061      for (x = 0; x < B; x++) {
062          *dst++ = *src++;
063      }
064
065      dst = x1.dp;
066      for (x = B; x < a->used; x++) {
067          *dst++ = *src++;
068      }
069  }
070
071  x0.used = B;
072  x1.used = a->used - B;

```

```

073
074     mp_clamp (&x0);
075
076     /* now calc the products x0*x0 and x1*x1 */
077     if (mp_sqr (&x0, &x0x0) != MP_OKAY)
078         goto X1X1;          /* x0x0 = x0*x0 */
079     if (mp_sqr (&x1, &x1x1) != MP_OKAY)
080         goto X1X1;          /* x1x1 = x1*x1 */
081
082     /* now calc (x1+x0)**2 */
083     if (s_mp_add (&x1, &x0, &t1) != MP_OKAY)
084         goto X1X1;          /* t1 = x1 - x0 */
085     if (mp_sqr (&t1, &t1) != MP_OKAY)
086         goto X1X1;          /* t1 = (x1 - x0) * (x1 - x0) */
087
088     /* add x0y0 */
089     if (s_mp_add (&x0x0, &x1x1, &t2) != MP_OKAY)
090         goto X1X1;          /* t2 = x0x0 + x1x1 */
091     if (s_mp_sub (&t1, &t2, &t1) != MP_OKAY)
092         goto X1X1;          /* t1 = (x1+x0)**2 - (x0x0 + x1x1) */
093
094     /* shift by B */
095     if (mp_lshd (&t1, B) != MP_OKAY)
096         goto X1X1;          /* t1 = (x0x0 + x1x1 - (x1-x0)*(x1-x0))<<B */
097     if (mp_lshd (&x1x1, B * 2) != MP_OKAY)
098         goto X1X1;          /* x1x1 = x1x1 << 2*B */
099
100     if (mp_add (&x0x0, &t1, &t1) != MP_OKAY)
101         goto X1X1;          /* t1 = x0x0 + t1 */
102     if (mp_add (&t1, &x1x1, b) != MP_OKAY)
103         goto X1X1;          /* t1 = x0x0 + t1 + x1x1 */
104
105     err = MP_OKAY;
106
107     X1X1:mp_clear (&x1x1);
108     X0X0:mp_clear (&x0x0);
109     T2:mp_clear (&t2);
110     T1:mp_clear (&t1);
111     X1:mp_clear (&x1);
112     X0:mp_clear (&x0);
113     ERR:
114     return err;

```

```

115  }
116  #endif
117
118  /* ref:          tag: v1.0.1 */
119  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
120  /* commit time:  2017-08-29 22:27:36 +0200 */

```

This implementation is largely based on the implementation of algorithm `mp_karatsuba_mul`. It uses the same inline style to copy and shift the input into the two halves. The loop from line 53 to line 69 has been modified since only one input exists. The **used** count of both $x0$ and $x1$ is fixed up and $x0$ is clamped before the calculations begin. At this point $x1$ and $x0$ are valid equivalents to the respective halves as if `mp_rshd` and `mp_mod_2d` had been used.

By inlining the copy and shift operations the cutoff point for Karatsuba multiplication can be lowered. On the Athlon the cutoff point is exactly at the point where Comba squaring can no longer be used (*128 digits*). On slower processors such as the Intel P4 it is actually below the Comba limit (*at 110 digits*).

This routine uses the same error trap coding style as `mp_karatsuba_sqr`. As the temporary variables are initialized errors are redirected to the error trap higher up. If the algorithm completes without error the error code is set to **MP_OKAY** and `mp_clears` are executed normally.

5.3.5 Toom-Cook Squaring

The Toom-Cook squaring algorithm `mp_toom_sqr` is heavily based on the algorithm `mp_toom_mul` with the exception that squarings are used instead of multiplication to find the five relations. The reader is encouraged to read the description of the latter algorithm and try to derive their own Toom-Cook squaring algorithm.

5.3.6 High Level Squaring

Algorithm **mp_sqr**.

Input. $\text{mp_int } a$

Output. $b \leftarrow a^2$

1. If $a.\text{used} \geq \text{TOOM_SQR_CUTOFF}$ then
 - 1.1 $b \leftarrow a^2$ using algorithm `mp_toom_sqr`
 2. else if $a.\text{used} \geq \text{KARATSUBA_SQR_CUTOFF}$ then
 - 2.1 $b \leftarrow a^2$ using algorithm `mp_karatsuba_sqr`
 3. else
 - 3.1 $\text{digs} \leftarrow a.\text{used} + b.\text{used} + 1$
 - 3.2 If $\text{digs} < \text{MP_ARRAY}$ and $a.\text{used} \leq \delta$ then
 - 3.2.1 $b \leftarrow a^2$ using algorithm `fast_s_mp_sqr`.
 - 3.3 else
 - 3.3.1 $b \leftarrow a^2$ using algorithm `s_mp_sqr`.
 4. $b.\text{sign} \leftarrow \text{MP_ZPOS}$
 5. Return the result of the unsigned squaring performed.
-

Figure 5.15: Algorithm `mp_sqr`

Algorithm `mp_sqr`. This algorithm computes the square of the input using one of four different algorithms. If the input is very large and has at least **TOOM_SQR_CUTOFF** or **KARATSUBA_SQR_CUTOFF** digits then either the Toom-Cook or the Karatsuba Squaring algorithm is used. If neither of the polynomial basis algorithms should be used then either the Comba or baseline algorithm is used.

File: `bn_mp_sqr.c`

```

016
017  /* computes b = a*a */
018  int
019  mp_sqr (mp_int * a, mp_int * b)
020  {
021      int      res;
022
023      #ifdef BN_MP_TOOM_SQR_C
024          /* use Toom-Cook? */
025          if (a->used >= TOOM_SQR_CUTOFF) {
026              res = mp_toom_sqr(a, b);
027              /* Karatsuba? */
028          } else
029      #endif

```

```

030 #ifdef BN_MP_KARATSUBA_SQR_C
031     if (a->used >= KARATSUBA_SQR_CUTOFF) {
032         res = mp_karatsuba_sqr (a, b);
033     } else
034 #endif
035     {
036 #ifdef BN_FAST_S_MP_SQR_C
037     /* can we use the fast comba multiplier? */
038     if (((a->used * 2) + 1) < MP_WARRAY) &&
039         (a->used <
040         (1 << (((sizeof(mp_word) * CHAR_BIT) - (2 * DIGIT_BIT)) - 1)))) {
041         res = fast_s_mp_sqr (a, b);
042     } else
043 #endif
044     {
045 #ifdef BN_S_MP_SQR_C
046         res = s_mp_sqr (a, b);
047 #else
048         res = MP_VAL;
049 #endif
050     }
051 }
052 b->sign = MP_ZPOS;
053 return res;
054 }
055 #endif
056
057 /* ref:          tag: v1.0.1 */
058 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
059 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Exercises

- [3] Devise an efficient algorithm for selection of the radix point to handle inputs that have different number of digits in Karatsuba multiplication.
- [2] In section 5.3 the fact that every column of a squaring is made up of double products and at most one square is stated. Prove this statement.
- [3] Prove the equation for Karatsuba squaring.
- [1] Prove that Karatsuba squaring requires $O(n^{\lg(3)})$ time.
- [2] Determine the minimal ratio between addition and multiplication clock cycles required for equation 6.7 to be true.
- [3] Implement a threaded version of Comba multiplication (and squaring) where you compute subsets of the columns in each thread. Determine a cutoff point where it is effective and add the logic to `mp_mul()` and `mp_sqr()`.
- [4] Same as the previous but also modify the Karatsuba and Toom-Cook. You must increase the throughput of `mp_exptmod()` for random odd moduli in the range 512...4096 bits significantly ($> 2x$) to complete this challenge.

Chapter 6

Modular Reduction

6.1 Basics of Modular Reduction

Modular reduction is an operation that arises quite often within public key cryptography algorithms and various number theoretic algorithms, such as factoring. Modular reduction algorithms are the third class of algorithms of the “multipliers” set. A number a is said to be *reduced* modulo another number b by finding the remainder of the division a/b . Full integer division with remainder is a topic to be covered in 8.1.

Modular reduction is equivalent to solving for r in the following equation. $a = bq + r$ where $q = \lfloor a/b \rfloor$. The result r is said to be “congruent to a modulo b ” which is also written as $r \equiv a \pmod{b}$. In other vernacular r is known as the “modular residue” which leads to “quadratic residue”¹ and other forms of residues.

Modular reductions are normally used to create either finite groups, rings or fields. The most common usage for performance driven modular reductions is in modular exponentiation algorithms. That is to compute $d = a^b \pmod{c}$ as fast as possible. This operation is used in the RSA and Diffie-Hellman public key algorithms, for example. Modular multiplication and squaring also appears as a fundamental operation in elliptic curve cryptographic algorithms. As will be discussed in the subsequent chapter there exist fast algorithms for computing modular exponentiations without having to perform (*in this example*) $b - 1$ multiplications. These algorithms will produce partial results in the

¹That’s fancy talk for $b \equiv a^2 \pmod{p}$.

range $0 \leq x < c^2$ which can be taken advantage of to create several efficient algorithms. They have also been used to create redundancy check algorithms known as CRCs, error correction codes such as Reed-Solomon and solve a variety of number theoretic problems.

6.2 The Barrett Reduction

The Barrett reduction algorithm [7] was inspired by fast division algorithms which multiply by the reciprocal to emulate division. Barretts observation was that the residue c of a modulo b is equal to

$$c = a - b \cdot \lfloor a/b \rfloor \quad (6.1)$$

Since algorithms such as modular exponentiation would be using the same modulus extensively, typical DSP² intuition would indicate the next step would be to replace a/b by a multiplication by the reciprocal. However, DSP intuition on its own will not work as these numbers are considerably larger than the precision of common DSP floating point data types. It would take another common optimization to optimize the algorithm.

6.2.1 Fixed Point Arithmetic

The trick used to optimize the above equation is based on a technique of emulating floating point data types with fixed precision integers. Fixed point arithmetic would become very popular as it greatly optimize the “3d-shooter” genre of games in the mid 1990s when floating point units were fairly slow if not unavailable. The idea behind fixed point arithmetic is to take a normal k -bit integer data type and break it into p -bit integer and a q -bit fraction part (*where* $p + q = k$).

In this system a k -bit integer n would actually represent $n/2^q$. For example, with $q = 4$ the integer $n = 37$ would actually represent the value 2.3125. To multiply two fixed point numbers the integers are multiplied using traditional arithmetic and subsequently normalized by moving the implied decimal point back to where it should be. For example, with $q = 4$ to multiply the integers 9 and 5 they must be converted to fixed point first by multiplying by 2^q . Let $a = 9(2^q)$ represent the fixed point representation of 9 and $b = 5(2^q)$ represent

²It is worth noting that Barrett’s paper targeted the DSP56K processor.

the fixed point representation of 5. The product ab is equal to $45(2^{2^q})$ which when normalized by dividing by 2^q produces $45(2^q)$.

This technique became popular since a normal integer multiplication and logical shift right are the only required operations to perform a multiplication of two fixed point numbers. Using fixed point arithmetic, division can be easily approximated by multiplying by the reciprocal. If 2^q is equivalent to one than $2^q/b$ is equivalent to the fixed point approximation of $1/b$ using real arithmetic. Using this fact dividing an integer a by another integer b can be achieved with the following expression.

$$\lfloor a/b \rfloor \sim \lfloor (a \cdot \lfloor 2^q/b \rfloor) / 2^q \rfloor \quad (6.2)$$

The precision of the division is proportional to the value of q . If the divisor b is used frequently as is the case with modular exponentiation pre-computing $2^q/b$ will allow a division to be performed with a multiplication and a right shift. Both operations are considerably faster than division on most processors.

Consider dividing 19 by 5. The correct result is $\lfloor 19/5 \rfloor = 3$. With $q = 3$ the reciprocal is $\lfloor 2^q/5 \rfloor = 1$ which leads to a product of 19 which when divided by 2^q produces 2. However, with $q = 4$ the reciprocal is $\lfloor 2^q/5 \rfloor = 3$ and the result of the emulated division is $\lfloor 3 \cdot 19/2^q \rfloor = 3$ which is correct. The value of 2^q must be close to or ideally larger than the dividend. In effect if a is the dividend then q should allow $0 \leq \lfloor a/2^q \rfloor \leq 1$ in order for this approach to work correctly. Plugging this form of division into the original equation the following modular residue equation arises.

$$c = a - b \cdot \lfloor (a \cdot \lfloor 2^q/b \rfloor) / 2^q \rfloor \quad (6.3)$$

Using the notation from [7] the value of $\lfloor 2^q/b \rfloor$ will be represented by the μ symbol. Using the μ variable also helps re-inforce the idea that it is meant to be computed once and re-used.

$$c = a - b \cdot \lfloor (a \cdot \mu) / 2^q \rfloor \quad (6.4)$$

Provided that $2^q \geq a$ this algorithm will produce a quotient that is either exactly correct or off by a value of one. In the context of Barrett reduction the value of a is bound by $0 \leq a \leq (b-1)^2$ meaning that $2^q \geq b^2$ is sufficient to ensure the reciprocal will have enough precision.

Let n represent the number of digits in b . This algorithm requires approximately $2n^2$ single precision multiplications to produce the quotient and another n^2 single precision multiplications to find the residue. In total $3n^2$ single precision multiplications are required to reduce the number.

For example, if $b = 1179677$ and $q = 41$ ($2^q > b^2$), then the reciprocal μ is equal to $\lfloor 2^q/b \rfloor = 1864089$. Consider reducing $a = 180388626447$ modulo b using the above reduction equation. The quotient using the new formula is $\lfloor (a \cdot \mu)/2^q \rfloor = 152913$. By subtracting $152913b$ from a the correct residue $a \equiv 677346 \pmod{b}$ is found.

6.2.2 Choosing a Radix Point

Using the fixed point representation a modular reduction can be performed with $3n^2$ single precision multiplications. If that were the best that could be achieved a full division³ might as well be used in its place. The key to optimizing the reduction is to reduce the precision of the initial multiplication that finds the quotient.

Let a represent the number of which the residue is sought. Let b represent the modulus used to find the residue. Let m represent the number of digits in b . For the purposes of this discussion we will assume that the number of digits in a is $2m$, which is generally true if two m -digit numbers have been multiplied. Dividing a by b is the same as dividing a $2m$ digit integer by a m digit integer. Digits below the $m - 1$ 'th digit of a will contribute at most a value of 1 to the quotient because $\beta^k < b$ for any $0 \leq k \leq m - 1$. Another way to express this is by re-writing a as two parts. If $a' \equiv a \pmod{b^m}$ and $a'' = a - a'$ then $\frac{a}{b} \equiv \frac{a' + a''}{b}$ which is equivalent to $\frac{a'}{b} + \frac{a''}{b}$. Since a' is bound to be less than b the quotient is bound by $0 \leq \frac{a'}{b} < 1$.

Since the digits of a' do not contribute much to the quotient the observation is that they might as well be zero. However, if the digits “might as well be zero” they might as well not be there in the first place. Let $q_0 = \lfloor a/\beta^{m-1} \rfloor$ represent the input with the irrelevant digits trimmed. Now the modular reduction is trimmed to the almost equivalent equation

$$c = a - b \cdot \lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor \quad (6.5)$$

Note that the original divisor 2^q has been replaced with β^{m+1} where in this case q is a multiple of $\lg(\beta)$. Also note that the exponent on the divisor when added to the amount q_0 was shifted by equals $2m$. If the optimization had not been performed the divisor would have the exponent $2m$ so in the end the exponents do “add up”. Using the above equation the quotient $\lfloor (q_0 \cdot \mu)/\beta^{m+1} \rfloor$ can be off from the true quotient by at most two. The original fixed point

³A division requires approximately $O(2cn^2)$ single precision multiplications for a small value of c . See 8.1 for further details.

quotient can be off by as much as one (*provided the radix point is chosen suitably*) and now that the lower irrelevant digits have been trimmed the quotient can be off by an additional value of one for a total of at most two. This implies that $0 \leq a - b \cdot \lfloor (q_0 \cdot \mu) / \beta^{m+1} \rfloor < 3b$. By first subtracting b times the quotient and then conditionally subtracting b once or twice the residue is found.

The quotient is now found using $(m+1)(m) = m^2 + m$ single precision multiplications and the residue with an additional m^2 single precision multiplications, ignoring the subtractions required. In total $2m^2 + m$ single precision multiplications are required to find the residue. This is considerably faster than the original attempt.

For example, let $\beta = 10$ represent the radix of the digits. Let $b = 9999$ represent the modulus which implies $m = 4$. Let $a = 99929878$ represent the value of which the residue is desired. In this case $q = 8$ since $10^7 < 9999^2$ meaning that $\mu = \lfloor \beta^q / b \rfloor = 10001$. With the new observation the multiplicand for the quotient is equal to $q_0 = \lfloor a / \beta^{m-1} \rfloor = 99929$. The quotient is then $\lfloor (q_0 \cdot \mu) / \beta^{m+1} \rfloor = 9993$. Subtracting $9993b$ from a and the correct residue $a \equiv 9871 \pmod{b}$ is found.

6.2.3 Trimming the Quotient

So far the reduction algorithm has been optimized from $3m^2$ single precision multiplications down to $2m^2 + m$ single precision multiplications. As it stands now the algorithm is already fairly fast compared to a full integer division algorithm. However, there is still room for optimization.

After the first multiplication inside the quotient ($q_0 \cdot \mu$) the value is shifted right by $m+1$ places effectively nullifying the lower half of the product. It would be nice to be able to remove those digits from the product to effectively cut down the number of single precision multiplications. If the number of digits in the modulus m is far less than β a full product is not required for the algorithm to work properly. In fact the lower $m-2$ digits will not affect the upper half of the product at all and do not need to be computed.

The value of μ is a m -digit number and q_0 is a $m+1$ digit number. Using a full multiplier $(m+1)(m) = m^2 + m$ single precision multiplications would be required. Using a multiplier that will only produce digits at and above the $m-1$ 'th digit reduces the number of single precision multiplications to $\frac{m^2+m}{2}$ single precision multiplications.

6.2.4 Trimming the Residue

After the quotient has been calculated it is used to reduce the input. As previously noted the algorithm is not exact and it can be off by a small multiple of the modulus, that is $0 \leq a - b \cdot \lfloor (q_0 \cdot \mu) / \beta^{m+1} \rfloor < 3b$. If b is m digits then the result of reduction equation is a value of at most $m + 1$ digits (*provided* $3 < \beta$) implying that the upper $m - 1$ digits are implicitly zero.

The next optimization arises from this very fact. Instead of computing $b \cdot \lfloor (q_0 \cdot \mu) / \beta^{m+1} \rfloor$ using a full $O(m^2)$ multiplication algorithm only the lower $m + 1$ digits of the product have to be computed. Similarly the value of a can be reduced modulo β^{m+1} before the multiple of b is subtracted which simplifies the subtraction as well. A multiplication that produces only the lower $m + 1$ digits requires $\frac{m^2 + 3m - 2}{2}$ single precision multiplications.

With both optimizations in place the algorithm is the algorithm Barrett proposed. It requires $m^2 + 2m - 1$ single precision multiplications which is considerably faster than the straightforward $3m^2$ method.

6.2.5 The Barrett Algorithm

Algorithm **mp_reduce**.

Input. mp_int a , mp_int b and $\mu = \lfloor \beta^{2m}/b \rfloor, m = \lceil \lg_\beta(b) \rceil, (0 \leq a < b^2, b > 1)$

Output. $a \pmod{b}$

Let m represent the number of digits in b .

1. Make a copy of a and store it in q . (*mp_init_copy*)
2. $q \leftarrow \lfloor q/\beta^{m-1} \rfloor$ (*mp_rshd*)

Produce the quotient.

3. $q \leftarrow q \cdot \mu$ (*note: only produce digits at or above $m-1$*)
4. $q \leftarrow \lfloor q/\beta^{m+1} \rfloor$

Subtract the multiple of modulus from the input.

5. $a \leftarrow a \pmod{\beta^{m+1}}$ (*mp_mod_2d*)
6. $q \leftarrow q \cdot b \pmod{\beta^{m+1}}$ (*s_mp_mul_digs*)
7. $a \leftarrow a - q$ (*mp_sub*)

Add β^{m+1} if a carry occurred.

8. If $a < 0$ then (*mp_cmp_d*)
 - 8.1 $q \leftarrow 1$ (*mp_set*)
 - 8.2 $q \leftarrow q \cdot \beta^{m+1}$ (*mp_lshd*)
 - 8.3 $a \leftarrow a + q$

Now subtract the modulus if the residue is too large (e.g. quotient too small).

9. While $a \geq b$ do (*mp_cmp*)
 - 9.1 $c \leftarrow a - b$
 10. Clear q .
 11. Return(*MP_OKAY*)
-

Figure 6.1: Algorithm mp_reduce

Algorithm mp_reduce. This algorithm will reduce the input a modulo b in place using the Barrett algorithm. It is loosely based on algorithm 14.42 of HAC [2, pp. 602] which is based on the paper from Paul Barrett [7]. The algorithm has several restrictions and assumptions which must be adhered to for the algorithm to work.

First the modulus b is assumed to be positive and greater than one. If the modulus were less than or equal to one then subtracting a multiple of it would either accomplish nothing or actually enlarge the input. The input a must be in the range $0 \leq a < b^2$ in order for the quotient to have enough precision. If a

is the product of two numbers that were already reduced modulo b , this will not be a problem. Technically the algorithm will still work if $a \geq b^2$ but it will take much longer to finish. The value of μ is passed as an argument to this algorithm and is assumed to be calculated and stored before the algorithm is used.

Recall that the multiplication for the quotient on step 3 must only produce digits at or above the $m-1$ 'th position. An algorithm called *s_mp_mul_high_digs* which has not been presented is used to accomplish this task. The algorithm is based on *s_mp_mul_digs* except that instead of stopping at a given level of precision it starts at a given level of precision. This optimal algorithm can only be used if the number of digits in b is very much smaller than β .

While it is known that $a \geq b \cdot \lfloor (q_0 \cdot \mu) / \beta^{m+1} \rfloor$ only the lower $m+1$ digits are being used to compute the residue, so an implied “borrow” from the higher digits might leave a negative result. After the multiple of the modulus has been subtracted from a the residue must be fixed up in case it is negative. The invariant β^{m+1} must be added to the residue to make it positive again.

The while loop at step 9 will subtract b until the residue is less than b . If the algorithm is performed correctly this step is performed at most twice, and on average once. However, if $a \geq b^2$ then it will iterate substantially more times than it should.

File: bn_mp_reduce.c

```

016
017  /* reduces x mod m, assumes 0 < x < m**2, mu is
018   * precomputed via mp_reduce_setup.
019   * From HAC pp.604 Algorithm 14.42
020   */
021  int mp_reduce (mp_int * x, mp_int * m, mp_int * mu)
022  {
023      mp_int  q;
024      int     res, um = m->used;
025
026      /* q = x */
027      if ((res = mp_init_copy (&q, x)) != MP_OKAY) {
028          return res;
029      }
030
031      /* q1 = x / b**(k-1) */
032      mp_rshd (&q, um - 1);
033
034      /* according to HAC this optimization is ok */
035      if (((mp_digit) um) > (((mp_digit)1) << (DIGIT_BIT - 1))) {

```

```

036         if ((res = mp_mul (&q, mu, &q)) != MP_OKAY) {
037             goto CLEANUP;
038         }
039     } else {
040 #ifdef BN_S_MP_MUL_HIGH_DIGS_C
041         if ((res = s_mp_mul_high_digs (&q, mu, &q, um)) != MP_OKAY) {
042             goto CLEANUP;
043         }
044 #elif defined(BN_FAST_S_MP_MUL_HIGH_DIGS_C)
045         if ((res = fast_s_mp_mul_high_digs (&q, mu, &q, um)) != MP_OKAY) {
046             goto CLEANUP;
047         }
048 #else
049         {
050             res = MP_VAL;
051             goto CLEANUP;
052         }
053 #endif
054     }
055
056     /* q3 = q2 / b**(k+1) */
057     mp_rshd (&q, um + 1);
058
059     /* x = x mod b**(k+1), quick (no division) */
060     if ((res = mp_mod_2d (x, DIGIT_BIT * (um + 1), x)) != MP_OKAY) {
061         goto CLEANUP;
062     }
063
064     /* q = q * m mod b**(k+1), quick (no division) */
065     if ((res = s_mp_mul_digs (&q, m, &q, um + 1)) != MP_OKAY) {
066         goto CLEANUP;
067     }
068
069     /* x = x - q */
070     if ((res = mp_sub (x, &q, x)) != MP_OKAY) {
071         goto CLEANUP;
072     }
073
074     /* If x < 0, add b**(k+1) to it */
075     if (mp_cmp_d (x, 0) == MP_LT) {
076         mp_set (&q, 1);
077         if ((res = mp_lshd (&q, um + 1)) != MP_OKAY)

```

```

078         goto CLEANUP;
079         if ((res = mp_add (x, &q, x)) != MP_OKAY)
080             goto CLEANUP;
081     }
082
083     /* Back off if it's too big */
084     while (mp_cmp (x, m) != MP_LT) {
085         if ((res = s_mp_sub (x, m, x)) != MP_OKAY) {
086             goto CLEANUP;
087         }
088     }
089
090 CLEANUP:
091     mp_clear (&q);
092
093     return res;
094 }
095 #endif
096
097 /* ref:          tag: v1.0.1 */
098 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
099 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The first multiplication that determines the quotient can be performed by only producing the digits from $m-1$ and up. This essentially halves the number of single precision multiplications required. However, the optimization is only safe if β is much larger than the number of digits in the modulus. In the source code this is evaluated on lines 36 to 43 where algorithm `s_mp_mul_high_digs` is used when it is safe to do so.

6.2.6 The Barrett Setup Algorithm

In order to use algorithm `mp_reduce` the value of μ must be calculated in advance. Ideally this value should be computed once and stored for future use so that the Barrett algorithm can be used without delay.

Algorithm **mp_reduce_setup**.
Input. mp_int a ($a > 1$)
Output. $\mu \leftarrow \lfloor \beta^{2^m} / a \rfloor$

1. $\mu \leftarrow 2^{2 \cdot \lg(\beta) \cdot m}$ (*mp_2expt*)
2. $\mu \leftarrow \lfloor \mu / b \rfloor$ (*mp_div*)
3. Return(*MP_OKAY*)

Figure 6.2: Algorithm mp_reduce_setup

Algorithm mp_reduce_setup. This algorithm computes the reciprocal μ required for Barrett reduction. First β^{2^m} is calculated as $2^{2 \cdot \lg(\beta) \cdot m}$ which is equivalent and much faster. The final value is computed by taking the integer quotient of $\lfloor \mu / b \rfloor$.

File: bn_mp_reduce_setup.c

```

016
017  /* pre-calculate the value required for Barrett reduction
018   * For a given modulus "b" it calculates the value required in "a"
019   */
020  int mp_reduce_setup (mp_int * a, mp_int * b)
021  {
022      int      res;
023
024      if ((res = mp_2expt (a, b->used * 2 * DIGIT_BIT)) != MP_OKAY) {
025          return res;
026      }
027      return mp_div (a, b, a, NULL);
028  }
029  #endif
030
031  /* ref:          tag: v1.0.1 */
032  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
033  /* commit time: 2017-08-29 22:27:36 +0200 */

```

This simple routine calculates the reciprocal μ required by Barrett reduction. Note the extended usage of algorithm mp_div where the variable which would received the remainder is passed as NULL. As will be discussed in 8.1 the division routine allows both the quotient and the remainder to be passed as NULL meaning to ignore the value.

6.3 The Montgomery Reduction

Montgomery reduction⁴ [8] is by far the most interesting form of reduction in common use. It computes a modular residue which is not actually equal to the residue of the input yet instead equal to a residue times a constant. However, as perplexing as this may sound the algorithm is relatively simple and very efficient.

Throughout this entire section the variable n will represent the modulus used to form the residue. As will be discussed shortly the value of n must be odd. The variable x will represent the quantity of which the residue is sought. Similar to the Barrett algorithm the input is restricted to $0 \leq x < n^2$. To begin the description some simple number theory facts must be established.

Fact 1. Adding n to x does not change the residue since in effect it adds one to the quotient $\lfloor x/n \rfloor$. Another way to explain this is that n is (*or multiples of n are*) congruent to zero modulo n . Adding zero will not change the value of the residue.

Fact 2. If x is even then performing a division by two in \mathbb{Z} is congruent to $x \cdot 2^{-1} \pmod{n}$. Actually this is an application of the fact that if x is evenly divisible by any $k \in \mathbb{Z}$ then division in \mathbb{Z} will be congruent to multiplication by k^{-1} modulo n .

From these two simple facts the following simple algorithm can be derived.

⁴Thanks to Niels Ferguson for his insightful explanation of the algorithm.

| |
|---|
| Algorithm Montgomery Reduction. |
| Input. Integer x , n and k |
| Output. $2^{-k}x \pmod{n}$ |

1. for t from 1 to k do
 - 1.1 If x is odd then
 - 1.1.1 $x \leftarrow x + n$
 - 1.2 $x \leftarrow x/2$
2. Return x .

Figure 6.3: Algorithm Montgomery Reduction

The algorithm reduces the input one bit at a time using the two congruencies stated previously. Inside the loop n , which is odd, is added to x if x is odd. This forces x to be even which allows the division by two in \mathbb{Z} to be congruent to a modular division by two. Since x is assumed to be initially much larger than n the addition of n will contribute an insignificant magnitude to x . Let r represent the final result of the Montgomery algorithm. If $k > \lg(n)$ and $0 \leq x < n^2$ then the final result is limited to $0 \leq r < \lfloor x/2^k \rfloor + n$. As a result at most a single subtraction is required to get the residue desired.

| Step number (t) | Result (x) |
|---------------------|----------------------------|
| 1 | $x + n = 5812, x/2 = 2906$ |
| 2 | $x/2 = 1453$ |
| 3 | $x + n = 1710, x/2 = 855$ |
| 4 | $x + n = 1112, x/2 = 556$ |
| 5 | $x/2 = 278$ |
| 6 | $x/2 = 139$ |
| 7 | $x + n = 396, x/2 = 198$ |
| 8 | $x/2 = 99$ |
| 9 | $x + n = 356, x/2 = 178$ |

Figure 6.4: Example of Montgomery Reduction (I)

Consider the example in figure 6.4 which reduces $x = 5555$ modulo $n = 257$ when $k = 9$ (note $\beta^k = 512$ which is larger than n). The result of the algorithm $r = 178$ is congruent to the value of $2^{-9} \cdot 5555 \pmod{257}$. When r is multiplied by 2^9 modulo 257 the correct residue $r \equiv 158$ is produced.

Let $k = \lfloor \lg(n) \rfloor + 1$ represent the number of bits in n . The current algorithm

requires $2k^2$ single precision shifts and k^2 single precision additions. At this rate the algorithm is most certainly slower than Barrett reduction and not terribly useful. Fortunately there exists an alternative representation of the algorithm.

| | |
|---|---------------------------------------|
| Algorithm Montgomery Reduction (modified I). | |
| Input. Integer x , n and k ($2^k > n$) | |
| Output. $2^{-k}x \pmod{n}$ | |
| <hr/> | |
| 1. | for t from 1 to k do |
| 1.1 | If the t 'th bit of x is one then |
| 1.1.1 | $x \leftarrow x + 2^t n$ |
| 2. | Return $x/2^k$. |

Figure 6.5: Algorithm Montgomery Reduction (modified I)

This algorithm is equivalent since $2^t n$ is a multiple of n and the lower k bits of x are zero by step 2. The number of single precision shifts has now been reduced from $2k^2$ to $k^2 + k$ which is only a small improvement.

| Step number (t) | Result (x) | Result (x) in Binary |
|---------------------|---------------------|--------------------------|
| – | 5555 | 1010110110011 |
| 1 | $x + 2^0 n = 5812$ | 1011010110100 |
| 2 | 5812 | 1011010110100 |
| 3 | $x + 2^2 n = 6840$ | 1101010111000 |
| 4 | $x + 2^3 n = 8896$ | 10001011000000 |
| 5 | 8896 | 10001011000000 |
| 6 | 8896 | 10001011000000 |
| 7 | $x + 2^6 n = 25344$ | 110001100000000 |
| 8 | 25344 | 110001100000000 |
| 9 | $x + 2^7 n = 91136$ | 10110010000000000 |
| – | $x/2^k = 178$ | |

Figure 6.6: Example of Montgomery Reduction (II)

Figure 6.6 demonstrates the modified algorithm reducing $x = 5555$ modulo $n = 257$ with $k = 9$. With this algorithm a single shift right at the end is the only right shift required to reduce the input instead of k right shifts inside the loop. Note that for the iterations $t = 2, 5, 6$ and 8 where the result x is not changed. In those iterations the t 'th bit of x is zero and the appropriate

multiple of n does not need to be added to force the t 'th bit of the result to zero.

6.3.1 Digit Based Montgomery Reduction

Instead of computing the reduction on a bit-by-bit basis it is actually much faster to compute it on digit-by-digit basis. Consider the previous algorithm re-written to compute the Montgomery reduction in this new fashion.

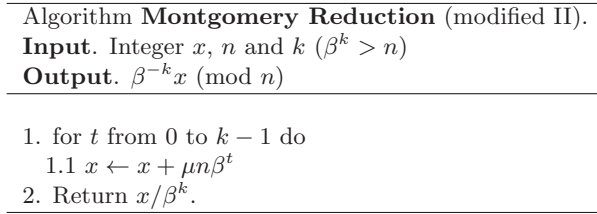


Figure 6.7: Algorithm Montgomery Reduction (modified II)

The value $\mu n \beta^t$ is a multiple of the modulus n meaning that it will not change the residue. If the first digit of the value $\mu n \beta^t$ equals the negative (modulo β) of the t 'th digit of x then the addition will result in a zero digit. This problem breaks down to solving the following congruency.

$$\begin{aligned} x_t + \mu n_0 &\equiv 0 \pmod{\beta} \\ \mu n_0 &\equiv -x_t \pmod{\beta} \\ \mu &\equiv -x_t/n_0 \pmod{\beta} \end{aligned}$$

In each iteration of the loop on step 1 a new value of μ must be calculated. The value of $-1/n_0 \pmod{\beta}$ is used extensively in this algorithm and should be precomputed. Let ρ represent the negative of the modular inverse of n_0 modulo β .

For example, let $\beta = 10$ represent the radix. Let $n = 17$ represent the modulus which implies $k = 2$ and $\rho \equiv 7$. Let $x = 33$ represent the value to reduce.

| Step (t) | Value of x | Value of μ |
|--------------|--------------------------|----------------|
| — | 33 | — |
| 0 | $33 + \mu n = 50$ | 1 |
| 1 | $50 + \mu n \beta = 900$ | 5 |

Figure 6.8: Example of Montgomery Reduction

The final result 900 is then divided by β^k to produce the final result 9. The first observation is that $9 \not\equiv x \pmod{n}$ which implies the result is not the modular residue of x modulo n . However, recall that the residue is actually multiplied by β^{-k} in the algorithm. To get the true residue the value must be multiplied by β^k . In this case $\beta^k \equiv 15 \pmod{n}$ and the correct residue is $9 \cdot 15 \equiv 16 \pmod{n}$.

6.3.2 Baseline Montgomery Reduction

The baseline Montgomery reduction algorithm will produce the residue for any size input. It is designed to be a catch-all algorithm for Montgomery reductions.

Algorithm **mp_montgomery_reduce**.

Input. mp_int x , mp_int n and a digit $\rho \equiv -1/n_0 \pmod{n}$.
 $(0 \leq x < n^2, n > 1, (n, \beta) = 1, \beta^k > n)$

Output. $\beta^{-k}x \pmod{n}$

1. $digs \leftarrow 2n.used + 1$
2. If $digs < MP_ARRAY$ and $m.used < \delta$ then
 - 2.1 Use algorithm fast_mp_montgomery_reduce instead.

Setup x for the reduction.

3. If $x.alloc < digs$ then grow x to $digs$ digits.
4. $x.used \leftarrow digs$

Eliminate the lower k digits.

5. For ix from 0 to $k - 1$ do
 - 5.1 $\mu \leftarrow x_{ix} \cdot \rho \pmod{\beta}$
 - 5.2 $u \leftarrow 0$
 - 5.3 For iy from 0 to $k - 1$ do
 - 5.3.1 $\hat{r} \leftarrow \mu n_{iy} + x_{ix+iy} + u$
 - 5.3.2 $x_{ix+iy} \leftarrow \hat{r} \pmod{\beta}$
 - 5.3.3 $u \leftarrow \lfloor \hat{r}/\beta \rfloor$
 - 5.4 While $u > 0$ do
 - 5.4.1 $iy \leftarrow iy + 1$
 - 5.4.2 $x_{ix+iy} \leftarrow x_{ix+iy} + u$
 - 5.4.3 $u \leftarrow \lfloor x_{ix+iy}/\beta \rfloor$
 - 5.4.4 $x_{ix+iy} \leftarrow x_{ix+iy} \pmod{\beta}$

Divide by β^k and fix up as required.

6. $x \leftarrow \lfloor x/\beta^k \rfloor$
 7. If $x \geq n$ then
 - 7.1 $x \leftarrow x - n$
 8. Return(MP_OKAY).
-

Figure 6.9: Algorithm mp_montgomery_reduce

Algorithm mp_montgomery_reduce. This algorithm reduces the input x modulo n in place using the Montgomery reduction algorithm. The algorithm is loosely based on algorithm 14.32 of [2, pp.601] except it merges the multiplication of $\mu n \beta^t$ with the addition in the inner loop. The restrictions on this algorithm are fairly easy to adapt to. First $0 \leq x < n^2$ bounds the input to

numbers in the same range as for the Barrett algorithm. Additionally if $n > 1$ and n is odd there will exist a modular inverse ρ . ρ must be calculated in advance of this algorithm. Finally the variable k is fixed and a pseudonym for $n.used$.

Step 2 decides whether a faster Montgomery algorithm can be used. It is based on the Comba technique meaning that there are limits on the size of the input. This algorithm is discussed in sub-section 6.3.3.

Step 5 is the main reduction loop of the algorithm. The value of μ is calculated once per iteration in the outer loop. The inner loop calculates $x + \mu n \beta^{ix}$ by multiplying μn and adding the result to x shifted by ix digits. Both the addition and multiplication are performed in the same loop to save time and memory. Step 5.4 will handle any additional carries that escape the inner loop.

Using a quick inspection this algorithm requires n single precision multiplications for the outer loop and n^2 single precision multiplications in the inner loop. In total $n^2 + n$ single precision multiplications which compares favourably to Barrett at $n^2 + 2n - 1$ single precision multiplications.

File: bn_mp_montgomery_reduce.c

```

016
017  /* computes xR**-1 == x (mod N) via Montgomery Reduction */
018  int
019  mp_montgomery_reduce (mp_int * x, mp_int * n, mp_digit rho)
020  {
021      int      ix, res, digs;
022      mp_digit mu;
023
024      /* can the fast reduction [comba] method be used?
025       *
026       * Note that unlike in mul you're safely allowed *less*
027       * than the available columns [255 per default] since carries
028       * are fixed up in the inner loop.
029       */
030      digs = (n->used * 2) + 1;
031      if ((digs < MP_WARRAY) &&
032          (n->used <
033           (1 << ((CHAR_BIT * sizeof(mp_word)) - (2 * DIGIT_BIT))))) {
034          return fast_mp_montgomery_reduce (x, n, rho);
035      }
036
037      /* grow the input as required */
038      if (x->alloc < digs) {

```

```

039     if ((res = mp_grow (x, digs)) != MP_OKAY) {
040         return res;
041     }
042 }
043 x->used = digs;
044
045 for (ix = 0; ix < n->used; ix++) {
046     /* mu = ai * rho mod b
047      *
048      * The value of rho must be precalculated via
049      * montgomery_setup() such that
050      * it equals -1/n0 mod b this allows the
051      * following inner loop to reduce the
052      * input one digit at a time
053      */
054     mu = (mp_digit) (((mp_word)x->dp[ix] * (mp_word)rho) & MP_MASK);
055
056     /* a = a + mu * m * b**i */
057     {
058         int iy;
059         mp_digit *tmpn, *tmpx, u;
060         mp_word r;
061
062         /* alias for digits of the modulus */
063         tmpn = n->dp;
064
065         /* alias for the digits of x [the input] */
066         tmpx = x->dp + ix;
067
068         /* set the carry to zero */
069         u = 0;
070
071         /* Multiply and add in place */
072         for (iy = 0; iy < n->used; iy++) {
073             /* compute product and sum */
074             r = ((mp_word)mu * (mp_word)*tmpn++) +
075                 (mp_word) u + (mp_word) *tmpx;
076
077             /* get carry */
078             u = (mp_digit)(r >> ((mp_word) DIGIT_BIT));
079
080             /* fix digit */

```

```

081         *tmpx++ = (mp_digit)(r & ((mp_word) MP_MASK));
082     }
083     /* At this point the ix'th digit of x should be zero */
084
085
086     /* propagate carries upwards as required*/
087     while (u != 0) {
088         *tmpx  += u;
089         u      = *tmpx >> DIGIT_BIT;
090         *tmpx++ &= MP_MASK;
091     }
092 }
093 }
094
095 /* at this point the n.used'th least
096  * significant digits of x are all zero
097  * which means we can shift x to the
098  * right by n.used digits and the
099  * residue is unchanged.
100  */
101
102 /* x = x/b**n.used */
103 mp_clamp(x);
104 mp_rshd (x, n->used);
105
106 /* if x >= n then x = x - n */
107 if (mp_cmp_mag (x, n) != MP_LT) {
108     return s_mp_sub (x, n, x);
109 }
110
111 return MP_OKAY;
112 }
113 #endif
114
115 /* ref:          tag: v1.0.1 */
116 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
117 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This is the baseline implementation of the Montgomery reduction algorithm. Lines 30 to 35 determine if the Comba based routine can be used instead. Line 48 computes the value of μ for that particular iteration of the outer loop.

The multiplication $\mu n \beta^{ix}$ is performed in one step in the inner loop. The

alias $tmpx$ refers to the ix 'th digit of x and the alias $tmpn$ refers to the modulus n .

6.3.3 Faster “Comba” Montgomery Reduction

The Montgomery reduction requires fewer single precision multiplications than a Barrett reduction, however it is much slower due to the serial nature of the inner loop. The Barrett reduction algorithm requires two slightly modified multipliers which can be implemented with the Comba technique. The Montgomery reduction algorithm cannot directly use the Comba technique to any significant advantage since the inner loop calculates a $k \times 1$ product k times.

The biggest obstacle is that at the ix 'th iteration of the outer loop the value of x_{ix} is required to calculate μ . This means the carries from 0 to $ix - 1$ must have been propagated upwards to form a valid ix 'th digit. The solution as it turns out is very simple. Perform a Comba like multiplier and inside the outer loop just after the inner loop fix up the $ix + 1$ 'th digit by forwarding the carry.

With this change in place the Montgomery reduction algorithm can be performed with a Comba style multiplication loop which substantially increases the speed of the algorithm.

Algorithm **fast_mp_montgomery_reduce**.

Input. mp_int x , mp_int n and a digit $\rho \equiv -1/n_0 \pmod{n}$.
 $(0 \leq x < n^2, n > 1, (n, \beta) = 1, \beta^k > n)$

Output. $\beta^{-k}x \pmod{n}$

Place an array of **MP_WARRAY** mp_word variables called \hat{W} on the stack.

1. if $x.alloc < n.used + 1$ then grow x to $n.used + 1$ digits.

Copy the digits of x into the array \hat{W}

2. For ix from 0 to $x.used - 1$ do

2.1 $\hat{W}_{ix} \leftarrow x_{ix}$

3. For ix from $x.used$ to $2n.used - 1$ do

3.1 $\hat{W}_{ix} \leftarrow 0$

Eliminate the lower k digits.

4. for ix from 0 to $n.used - 1$ do

4.1 $\mu \leftarrow \hat{W}_{ix} \cdot \rho \pmod{\beta}$

4.2 For iy from 0 to $n.used - 1$ do

4.2.1 $\hat{W}_{iy+ix} \leftarrow \hat{W}_{iy+ix} + \mu \cdot n_{iy}$

4.3 $\hat{W}_{ix+1} \leftarrow \hat{W}_{ix+1} + \lfloor \hat{W}_{ix}/\beta \rfloor$

Propagate carries upwards.

5. for ix from $n.used$ to $2n.used + 1$ do

5.1 $\hat{W}_{ix+1} \leftarrow \hat{W}_{ix+1} + \lfloor \hat{W}_{ix}/\beta \rfloor$

Shift right and reduce modulo β simultaneously.

6. for ix from 0 to $n.used + 1$ do

6.1 $x_{ix} \leftarrow \hat{W}_{ix+n.used} \pmod{\beta}$

Zero excess digits and fixup x .

7. if $x.used > n.used + 1$ then do

7.1 for ix from $n.used + 1$ to $x.used - 1$ do

7.1.1 $x_{ix} \leftarrow 0$

8. $x.used \leftarrow n.used + 1$

9. Clamp excessive digits of x .

10. If $x \geq n$ then

10.1 $x \leftarrow x - n$

11. Return(**MP_OKAY**).

Figure 6.10: Algorithm fast_mp_montgomery_reduce

Algorithm fast_mp_montgomery_reduce. This algorithm will compute the Montgomery reduction of x modulo n using the Comba technique. It is on most computer platforms significantly faster than algorithm mp_montgomery_reduce and algorithm mp_reduce (*Barrett reduction*). The algorithm has the same re-

strictions on the input as the baseline reduction algorithm. An additional two restrictions are imposed on this algorithm. The number of digits k in the modulus n must not violate $MP_WARRAY > 2k+1$ and $n < \delta$. When $\beta = 2^{28}$ this algorithm can be used to reduce modulo a modulus of at most 3,556 bits in length.

As in the other Comba reduction algorithms there is a \hat{W} array which stores the columns of the product. It is initially filled with the contents of x with the excess digits zeroed. The reduction loop is very similar to the baseline loop at heart. The multiplication on step 4.1 can be single precision only since $ab \pmod{\beta} \equiv (a \pmod{\beta})(b \pmod{\beta})$. Some multipliers such as those on the ARM processors take a variable length time to complete depending on the number of bytes of result it must produce. By performing a single precision multiplication instead half the amount of time is spent.

Also note that digit \hat{W}_{ix} must have the carry from the $ix-1$ 'th digit propagated upwards in order for this to work. That is what step 4.3 will do. In effect over the n_used iterations of the outer loop the n_used 'th lower columns all have their carries propagated forwards. Note how the upper bits of those same words are not reduced modulo β . This is because those values will be discarded shortly and there is no point.

Step 5 will propagate the remainder of the carries upwards. On step 6 the columns are reduced modulo β and shifted simultaneously as they are stored in the destination x .

File: bn_fast_mp_montgomery_reduce.c

```

016
017  /* computes xR**(-1) == x (mod N) via Montgomery Reduction
018  *
019  * This is an optimized implementation of montgomery_reduce
020  * which uses the comba method to quickly calculate the columns of the
021  * reduction.
022  *
023  * Based on Algorithm 14.32 on pp.601 of HAC.
024  */
025  int fast_mp_montgomery_reduce (mp_int * x, mp_int * n, mp_digit rho)
026  {
027      int      ix, res, olduse;
028      mp_word W[MP_WARRAY];
029
030      /* get old used count */
031      olduse = x->used;
032

```

```

033  /* grow a as required */
034  if (x->alloc < (n->used + 1)) {
035      if ((res = mp_grow (x, n->used + 1)) != MP_OKAY) {
036          return res;
037      }
038  }
039
040  /* first we have to get the digits of the input into
041   * an array of double precision words W[...]
042   */
043  {
044      mp_word *_W;
045      mp_digit *tmpx;
046
047      /* alias for the W[] array */
048      _W = W;
049
050      /* alias for the digits of x */
051      tmpx = x->dp;
052
053      /* copy the digits of a into W[0..a->used-1] */
054      for (ix = 0; ix < x->used; ix++) {
055          *_W++ = *tmpx++;
056      }
057
058      /* zero the high words of W[a->used..m->used*2] */
059      for (; ix < ((n->used * 2) + 1); ix++) {
060          *_W++ = 0;
061      }
062  }
063
064  /* now we proceed to zero successive digits
065   * from the least significant upwards
066   */
067  for (ix = 0; ix < n->used; ix++) {
068      /* mu = ai * m' mod b
069       *
070       * We avoid a double precision multiplication (which isn't required)
071       * by casting the value down to a mp_digit. Note this requires
072       * that W[ix-1] have the carry cleared (see after the inner loop)
073       */
074      mp_digit mu;

```

```

075     mu = (mp_digit) (((W[ix] & MP_MASK) * rho) & MP_MASK);
076
077     /* a = a + mu * m * b**i
078     *
079     * This is computed in place and on the fly. The multiplication
080     * by b**i is handled by offseting which columns the results
081     * are added to.
082     *
083     * Note the comba method normally doesn't handle carries in the
084     * inner loop In this case we fix the carry from the previous
085     * column since the Montgomery reduction requires digits of the
086     * result (so far) [see above] to work. This is
087     * handled by fixing up one carry after the inner loop. The
088     * carry fixups are done in order so after these loops the
089     * first m->used words of W[] have the carries fixed
090     */
091     {
092         int iy;
093         mp_digit *tmpn;
094         mp_word *_W;
095
096         /* alias for the digits of the modulus */
097         tmpn = n->dp;
098
099         /* Alias for the columns set by an offset of ix */
100         _W = W + ix;
101
102         /* inner loop */
103         for (iy = 0; iy < n->used; iy++) {
104             *_W++ += ((mp_word)mu) * ((mp_word)*tmpn++);
105         }
106     }
107
108     /* now fix carry for next digit, W[ix+1] */
109     W[ix + 1] += W[ix] >> ((mp_word) DIGIT_BIT);
110 }
111
112 /* now we have to propagate the carries and
113 * shift the words downward [all those least
114 * significant digits we zeroed].
115 */
116 {

```

```

117     mp_digit *tmpx;
118     mp_word *_W, *_W1;
119
120     /* nox fix rest of carries */
121
122     /* alias for current word */
123     _W1 = W + ix;
124
125     /* alias for next word, where the carry goes */
126     _W = W + ++ix;
127
128     for (; ix <= ((n->used * 2) + 1); ix++) {
129         *_W++ += *_W1++ >> ((mp_word) DIGIT_BIT);
130     }
131
132     /* copy out, A = A/b**n
133     *
134     * The result is A/b**n but instead of converting from an
135     * array of mp_word to mp_digit than calling mp_rshd
136     * we just copy them in the right order
137     */
138
139     /* alias for destination word */
140     tmpx = x->dp;
141
142     /* alias for shifted double precision result */
143     _W = W + n->used;
144
145     for (ix = 0; ix < (n->used + 1); ix++) {
146         *tmpx++ = (mp_digit)(*_W++ & ((mp_word) MP_MASK));
147     }
148
149     /* zero oldused digits, if the input a was larger than
150     * m->used+1 we'll have to clear the digits
151     */
152     for (; ix < olduse; ix++) {
153         *tmpx++ = 0;
154     }
155 }
156
157 /* set the max used and clamp */
158 x->used = n->used + 1;

```

```

159     mp_clamp (x);
160
161     /* if A >= m then A = A - m */
162     if (mp_cmp_mag (x, n) != MP_LT) {
163         return s_mp_sub (x, n, x);
164     }
165     return MP_OKAY;
166 }
167 #endif
168
169 /* ref:          tag: v1.0.1 */
170 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
171 /* commit time:  2017-08-29 22:27:36 +0200 */

```

The \hat{W} array is first filled with digits of x on line 50 then the rest of the digits are zeroed on line 54. Both loops share the same alias variables to make the code easier to read.

The value of μ is calculated in an interesting fashion. First the value \hat{W}_{ix} is reduced modulo β and cast to a `mp_digit`. This forces the compiler to use a single precision multiplication and prevents any concerns about loss of precision. Line 109 fixes the carry for the next iteration of the loop by propagating the carry from \hat{W}_{ix} to \hat{W}_{ix+1} .

The for loop on line 108 propagates the rest of the carries upwards through the columns. The for loop on line 125 reduces the columns modulo β and shifts them k places at the same time. The alias $_{\hat{W}}$ actually refers to the array \hat{W} starting at the n_{used} 'th digit, that is $_{\hat{W}}_t = \hat{W}_{n_{\text{used}}+t}$.

6.3.4 Montgomery Setup

To calculate the variable ρ a relatively simple algorithm will be required.

Algorithm `mp_montgomery_setup`. This algorithm will calculate the value of ρ required within the Montgomery reduction algorithms. It uses a very interesting trick to calculate $1/n_0$ when β is a power of two.

File: `bn_mp_montgomery_setup.c`

```

016
017 /* setups the montgomery reduction stuff */
018 int
019 mp_montgomery_setup (mp_int * n, mp_digit * rho)
020 {
021     mp_digit x, b;

```

Algorithm **mp_montgomery_setup**.
Input. mp_int n ($n > 1$ and $(n, 2) = 1$)
Output. $\rho \equiv -1/n_0 \pmod{\beta}$

1. $b \leftarrow n_0$
2. If b is even return(*MP_VAL*)
3. $x \leftarrow (((b + 2) \text{ AND } 4) << 1) + b$
4. for k from 0 to $\lceil \lg(\lg(\beta)) \rceil - 2$ do
 - 4.1 $x \leftarrow x \cdot (2 - bx)$
5. $\rho \leftarrow \beta - x \pmod{\beta}$
6. Return(*MP_OKAY*).

Figure 6.11: Algorithm mp_montgomery_setup

```

022
023 /* fast inversion mod 2**k
024  *
025  * Based on the fact that
026  *
027  * XA = 1 (mod 2**n)  =>  (X(2-XA)) A = 1 (mod 2**2n)
028  *                    =>  2*XA - X*XA*A = 1
029  *                    =>  2*(1) - (1)      = 1
030  */
031  b = n->dp[0];
032
033  if ((b & 1) == 0) {
034      return MP_VAL;
035  }
036
037  x = (((b + 2) & 4) << 1) + b; /* here x*a==1 mod 2**4 */
038  x *= 2 - (b * x);             /* here x*a==1 mod 2**8 */
039  #if !defined(MP_8BIT)
040      x *= 2 - (b * x);          /* here x*a==1 mod 2**16 */
041  #endif
042  #if defined(MP_64BIT) || !(defined(MP_8BIT) || defined(MP_16BIT))
043      x *= 2 - (b * x);          /* here x*a==1 mod 2**32 */
044  #endif
045  #ifdef MP_64BIT
046      x *= 2 - (b * x);          /* here x*a==1 mod 2**64 */
047  #endif

```



```

048
049     /* rho = -1/m mod b */
050     *rho = (mp_digit)((mp_word)1 << ((mp_word) DIGIT_BIT)) - x) & MP_MASK;
051
052     return MP_OKAY;
053 }
054 #endif
055
056 /* ref:          tag: v1.0.1 */
057 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
058 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This source code computes the value of ρ required to perform Montgomery reduction. It has been modified to avoid performing excess multiplications when β is not the default 28-bits.

6.4 The Diminished Radix Algorithm

The Diminished Radix method of modular reduction [9] is a fairly clever technique which can be more efficient than either the Barrett or Montgomery methods for certain forms of moduli. The technique is based on the following simple congruence.

$$(x \bmod n) + k \lfloor x/n \rfloor \equiv x \pmod{(n - k)} \quad (6.6)$$

This observation was used in the MMB [10] block cipher to create a diffusion primitive. It used the fact that if $n = 2^{31}$ and $k = 1$ that then a x86 multiplier could produce the 62-bit product and use the “shrd” instruction to perform a double-precision right shift. The proof of the above equation is very simple. First write x in the product form.

$$x = qn + r \quad (6.7)$$

Now reduce both sides modulo $(n - k)$.

$$x \equiv qk + r \pmod{(n - k)} \quad (6.8)$$

The variable n reduces modulo $n - k$ to k . By putting $q = \lfloor x/n \rfloor$ and $r = x \bmod n$ into the equation the original congruence is reproduced, thus concluding the proof. The following algorithm is based on this observation.

Algorithm Diminished Radix Reduction.
Input. Integer x, n, k
Output. $x \bmod (n - k)$

1. $q \leftarrow \lfloor x/n \rfloor$
 2. $q \leftarrow k \cdot q$
 3. $x \leftarrow x \bmod n$
 4. $x \leftarrow x + q$
 5. If $x \geq (n - k)$ then
 - 5.1 $x \leftarrow x - (n - k)$
 - 5.2 Goto step 1.
 6. Return x
-

Figure 6.12: Algorithm Diminished Radix Reduction

This algorithm will reduce x modulo $n - k$ and return the residue. If $0 \leq x < (n - k)^2$ then the algorithm will loop almost always once or twice and occasionally three times. For simplicity sake the value of x is bounded by the following simple polynomial.

$$0 \leq x < n^2 + k^2 - 2nk \quad (6.9)$$

The true bound is $0 \leq x < (n - k - 1)^2$ but this has quite a few more terms. The value of q after step 1 is bounded by the following.

$$q < n - 2k - k^2/n \quad (6.10)$$

Since k^2 is going to be considerably smaller than n that term will always be zero. The value of x after step 3 is bounded trivially as $0 \leq x < n$. By step four the sum $x + q$ is bounded by

$$0 \leq q + x < (k + 1)n - 2k^2 - 1 \quad (6.11)$$

With a second pass q will be loosely bounded by $0 \leq q < k^2$ after step 2 while x will still be loosely bounded by $0 \leq x < n$ after step 3. After the second pass it is highly unlikely that the sum in step 4 will exceed $n - k$. In practice fewer than three passes of the algorithm are required to reduce virtually every input in the range $0 \leq x < (n - k - 1)^2$.

Figure 6.13 demonstrates the reduction of $x = 123456789$ modulo $n - k = 253$ when $n = 256$ and $k = 3$. Note that even while x is considerably larger than

| |
|---|
| $x = 123456789, n = 256, k = 3$ |
| $q \leftarrow \lfloor x/n \rfloor = 482253$ |
| $q \leftarrow q * k = 1446759$ |
| $x \leftarrow x \bmod n = 21$ |
| $x \leftarrow x + q = 1446780$ |
| $x \leftarrow x - (n - k) = 1446527$ |
| $q \leftarrow \lfloor x/n \rfloor = 5650$ |
| $q \leftarrow q * k = 16950$ |
| $x \leftarrow x \bmod n = 127$ |
| $x \leftarrow x + q = 17077$ |
| $x \leftarrow x - (n - k) = 16824$ |
| $q \leftarrow \lfloor x/n \rfloor = 65$ |
| $q \leftarrow q * k = 195$ |
| $x \leftarrow x \bmod n = 184$ |
| $x \leftarrow x + q = 379$ |
| $x \leftarrow x - (n - k) = 126$ |

Figure 6.13: Example Diminished Radix Reduction

$(n - k - 1)^2 = 63504$ the algorithm still converges on the modular residue exceedingly fast. In this case only three passes were required to find the residue $x \equiv 126$.

6.4.1 Choice of Moduli

On the surface this algorithm looks like a very expensive algorithm. It requires a couple of subtractions followed by multiplication and other modular reductions. The usefulness of this algorithm becomes exceedingly clear when an appropriate modulus is chosen.

Division in general is a very expensive operation to perform. The one exception is when the division is by a power of the radix of representation used. Division by ten for example is simple for pencil and paper mathematics since it amounts to shifting the decimal place to the right. Similarly division by two (*or powers of two*) is very simple for binary computers to perform. It would therefore seem logical to choose n of the form 2^p which would imply that $\lfloor x/n \rfloor$ is a simple shift of x right p bits.

However, there is one operation related to division of power of twos that is even faster than this. If $n = \beta^p$ then the division may be performed by moving whole digits to the right p places. In practice division by β^p is much faster than

division by 2^p for any p . Also with the choice of $n = \beta^p$ reducing x modulo n merely requires zeroing the digits above the $p - 1$ 'th digit of x .

Throughout the next section the term “restricted modulus” will refer to a modulus of the form $\beta^p - k$ whereas the term “unrestricted modulus” will refer to a modulus of the form $2^p - k$. The word “restricted” in this case refers to the fact that it is based on the 2^p logic except p must be a multiple of $\lg(\beta)$.

6.4.2 Choice of k

Now that division and reduction (*step 1 and 3 of figure 6.12*) have been optimized to simple digit operations the multiplication by k in step 2 is the most expensive operation. Fortunately the choice of k is not terribly limited. For all intents and purposes it might as well be a single digit. The smaller the value of k is the faster the algorithm will be.

6.4.3 Restricted Diminished Radix Reduction

The restricted Diminished Radix algorithm can quickly reduce an input modulo a modulus of the form $n = \beta^p - k$. This algorithm can reduce an input x within the range $0 \leq x < n^2$ using only a couple passes of the algorithm demonstrated in figure 6.12. The implementation of this algorithm has been optimized to avoid additional overhead associated with a division by β^p , the multiplication by k or the addition of x and q . The resulting algorithm is very efficient and can lead to substantial improvements over Barrett and Montgomery reduction when modular exponentiations are performed.

Algorithm **mp_dr_reduce**.
Input. mp_int x , n and a mp_digit $k = \beta - n_0$
 $(0 \leq x < n^2, n > 1, 0 < k < \beta)$
Output. $x \bmod n$

1. $m \leftarrow n.\text{used}$
2. If $x.\text{alloc} < 2m$ then grow x to $2m$ digits.
3. $\mu \leftarrow 0$
4. for i from 0 to $m - 1$ do
 - 4.1 $\hat{r} \leftarrow k \cdot x_{m+i} + x_i + \mu$
 - 4.2 $x_i \leftarrow \hat{r} \bmod \beta$
 - 4.3 $\mu \leftarrow \lfloor \hat{r} / \beta \rfloor$
5. $x_m \leftarrow \mu$
6. for i from $m + 1$ to $x.\text{used} - 1$ do
 - 6.1 $x_i \leftarrow 0$
7. Clamp excess digits of x .
8. If $x \geq n$ then
 - 8.1 $x \leftarrow x - n$
 - 8.2 Goto step 3.
9. Return(*MP_OKAY*).

Figure 6.14: Algorithm mp_dr_reduce

Algorithm mp_dr_reduce. This algorithm will perform the Diminished Radix reduction of x modulo n . It has similar restrictions to that of the Barrett reduction with the addition that n must be of the form $n = \beta^m - k$ where $0 < k < \beta$.

This algorithm essentially implements the pseudo-code in figure 6.12 except with a slight optimization. The division by β^m , multiplication by k and addition of $x \bmod \beta^m$ are all performed simultaneously inside the loop on step 4. The division by β^m is emulated by accessing the term at the $m + i$ 'th position which is subsequently multiplied by k and added to the term at the i 'th position. After the loop the m 'th digit is set to the carry and the upper digits are zeroed. Steps 5 and 6 emulate the reduction modulo β^m that should have happened to x before the addition of the multiple of the upper half.

At step 8 if x is still larger than n another pass of the algorithm is required. First n is subtracted from x and then the algorithm resumes at step 3.

File: bn_mp_dr_reduce.c

016

017 /* reduce "x" in place modulo "n" using the Diminished Radix algorithm.

```

018  *
019  * Based on algorithm from the paper
020  *
021  * "Generating Efficient Primes for Discrete Log Cryptosystems"
022  *           Chae Hoon Lim, Pil Joong Lee,
023  *           POSTECH Information Research Laboratories
024  *
025  * The modulus must be of a special format [see manual]
026  *
027  * Has been modified to use algorithm 7.10 from the LTM book instead
028  *
029  * Input x must be in the range  $0 \leq x \leq (n-1)**2$ 
030  */
031  int
032  mp_dr_reduce (mp_int * x, mp_int * n, mp_digit k)
033  {
034      int      err, i, m;
035      mp_word  r;
036      mp_digit mu, *tmpx1, *tmpx2;
037
038      /* m = digits in modulus */
039      m = n->used;
040
041      /* ensure that "x" has at least 2m digits */
042      if (x->alloc < (m + m)) {
043          if ((err = mp_grow (x, m + m)) != MP_OKAY) {
044              return err;
045          }
046      }
047
048      /* top of loop, this is where the code resumes if
049       * another reduction pass is required.
050       */
051      top:
052      /* aliases for digits */
053      /* alias for lower half of x */
054      tmpx1 = x->dp;
055
056      /* alias for upper half of x, or  $x/B^{**m}$  */
057      tmpx2 = x->dp + m;
058
059      /* set carry to zero */

```

```

060     mu = 0;
061
062     /* compute (x mod B**m) + k * [x/B**m] inline and inplace */
063     for (i = 0; i < m; i++) {
064         r          = ((mp_word)*tmpx2++) * (mp_word)k) + *tmpx1 + mu;
065         *tmpx1++   = (mp_digit)(r & MP_MASK);
066         mu         = (mp_digit)(r >> ((mp_word)DIGIT_BIT));
067     }
068
069     /* set final carry */
070     *tmpx1++ = mu;
071
072     /* zero words above m */
073     for (i = m + 1; i < x->used; i++) {
074         *tmpx1++ = 0;
075     }
076
077     /* clamp, sub and return */
078     mp_clamp (x);
079
080     /* if x >= n then subtract and reduce again
081      * Each successive "recursion" makes the input smaller and smaller.
082      */
083     if (mp_cmp_mag (x, n) != MP_LT) {
084         if ((err = s_mp_sub(x, n, x)) != MP_OKAY) {
085             return err;
086         }
087         goto top;
088     }
089     return MP_OKAY;
090 }
091 #endif
092
093 /* ref:          tag: v1.0.1 */
094 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
095 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The first step is to grow x as required to $2m$ digits since the reduction is performed in place on x . The label on line 51 is where the algorithm will resume if further reduction passes are required. In theory it could be placed at the top of the function however, the size of the modulus and question of whether x is large enough are invariant after the first pass meaning that it would be a waste

of time.

The aliases *tmpx1* and *tmpx2* refer to the digits of x where the latter is offset by m digits. By reading digits from x offset by m digits a division by β^m can be simulated virtually for free. The loop on line 63 performs the bulk of the work (*corresponds to step 4 of algorithm 7.11*) in this algorithm.

By line 70 the pointer *tmpx1* points to the m 'th digit of x which is where the final carry will be placed. Similarly by line 73 the same pointer will point to the $m + 1$ 'th digit where the zeroes will be placed.

Since the algorithm is only valid if both x and n are greater than zero an unsigned comparison suffices to determine if another pass is required. With the same logic at line 84 the value of x is known to be greater than or equal to n meaning that an unsigned subtraction can be used as well. Since the destination of the subtraction is the larger of the inputs the call to algorithm `s_mp_sub` cannot fail and the return code does not need to be checked.

Setup

To setup the restricted Diminished Radix algorithm the value $k = \beta - n_0$ is required. This algorithm is not really complicated but provided for completeness.

Algorithm **mp_dr_setup**.
Input. mp_int n
Output. $k = \beta - n_0$

1. $k \leftarrow \beta - n_0$

Figure 6.15: Algorithm `mp_dr_setup`

File: `bn_mp_dr_setup.c`

```

016
017  /* determines the setup value */
018  void mp_dr_setup(mp_int *a, mp_digit *d)
019  {
020      /* the casts are required if DIGIT_BIT is one less than
021       * the number of bits in a mp_digit [e.g. DIGIT_BIT==31]
022       */
023      *d = (mp_digit)((((mp_word)1) << ((mp_word)DIGIT_BIT)) -
024                  ((mp_word)a->dp[0]));
025  }
```



```

026
027  #endif
028
029  /* ref:          tag: v1.0.1 */
030  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
031  /* commit time:  2017-08-29 22:27:36 +0200 */

```

Modulus Detection

Another algorithm which will be useful is the ability to detect a restricted Diminished Radix modulus. An integer is said to be of restricted Diminished Radix form if all of the digits are equal to $\beta - 1$ except the trailing digit which may be any value.

Algorithm **mp_dr_is_modulus**.
Input. mp_int n
Output. 1 if n is in D.R form, 0 otherwise

1. If $n.used < 2$ then return(0).
2. for ix from 1 to $n.used - 1$ do
 - 2.1 If $n_{ix} \neq \beta - 1$ return(0).
3. Return(1).

Figure 6.16: Algorithm mp_dr_is_modulus

Algorithm mp_dr_is_modulus. This algorithm determines if a value is in Diminished Radix form. Step 1 rejects obvious cases where fewer than two digits are in the mp_int. Step 2 tests all but the first digit to see if they are equal to $\beta - 1$. If the algorithm manages to get to step 3 then n must be of Diminished Radix form.

File: bn_mp_dr_is_modulus.c

```

016
017  /* determines if a number is a valid DR modulus */
018  int mp_dr_is_modulus(mp_int *a)
019  {
020      int ix;
021
022      /* must be at least two digits */
023      if (a->used < 2) {
024          return 0;
025      }

```

```

026
027     /* must be of the form b**k - a [a <= b] so all
028     * but the first digit must be equal to -1 (mod b).
029     */
030     for (ix = 1; ix < a->used; ix++) {
031         if (a->dp[ix] != MP_MASK) {
032             return 0;
033         }
034     }
035     return 1;
036 }
037
038 #endif
039
040 /* ref:          tag: v1.0.1 */
041 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
042 /* commit time: 2017-08-29 22:27:36 +0200 */

```

6.4.4 Unrestricted Diminished Radix Reduction

The unrestricted Diminished Radix algorithm allows modular reductions to be performed when the modulus is of the form $2^p - k$. This algorithm is a straightforward adaptation of algorithm 6.12.

In general the restricted Diminished Radix reduction algorithm is much faster since it has considerably lower overhead. However, this new algorithm is much faster than either Montgomery or Barrett reduction when the moduli are of the appropriate form.

Algorithm mp_reduce_2k. This algorithm quickly reduces an input a modulo an unrestricted Diminished Radix modulus n . Division by 2^p is emulated with a right shift which makes the algorithm fairly inexpensive to use.

File: bn_mp_reduce_2k.c

```

016
017 /* reduces a modulo n where n is of the form 2**p - d */
018 int mp_reduce_2k(mp_int *a, mp_int *n, mp_digit d)
019 {
020     mp_int q;
021     int    p, res;
022
023     if ((res = mp_init(&q)) != MP_OKAY) {
024         return res;

```

Algorithm **mp_reduce_2k**.
Input. mp_int a and n . mp_digit k
 $(a \geq 0, n > 1, 0 < k < \beta, n + k \text{ is a power of two})$
Output. $a \pmod{n}$

1. $p \leftarrow \lceil \lg(n) \rceil$ (*mp_count_bits*)
2. While $a \geq n$ do
 - 2.1 $q \leftarrow \lfloor a/2^p \rfloor$ (*mp_div_2d*)
 - 2.2 $a \leftarrow a \pmod{2^p}$ (*mp_mod_2d*)
 - 2.3 $q \leftarrow q \cdot k$ (*mp_mul_d*)
 - 2.4 $a \leftarrow a - q$ (*s_mp_sub*)
 - 2.5 If $a \geq n$ then do
 - 2.5.1 $a \leftarrow a - n$
3. Return(*MP_OKAY*).

Figure 6.17: Algorithm mp_reduce_2k

```

025     }
026
027     p = mp_count_bits(n);
028 top:
029     /* q = a/2**p, a = a mod 2**p */
030     if ((res = mp_div_2d(a, p, &q, a)) != MP_OKAY) {
031         goto ERR;
032     }
033
034     if (d != 1) {
035         /* q = q * d */
036         if ((res = mp_mul_d(&q, d, &q)) != MP_OKAY) {
037             goto ERR;
038         }
039     }
040
041     /* a = a + q */
042     if ((res = s_mp_add(a, &q, a)) != MP_OKAY) {
043         goto ERR;
044     }
045
046     if (mp_cmp_mag(a, n) != MP_LT) {
047         if ((res = s_mp_sub(a, n, a)) != MP_OKAY) {
048             goto ERR;

```

```

049         }
050         goto top;
051     }
052
053     ERR:
054         mp_clear(&q);
055         return res;
056     }
057
058 #endif
059
060 /* ref:          tag: v1.0.1 */
061 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
062 /* commit time: 2017-08-29 22:27:36 +0200 */

```

The algorithm `mp_count_bits` calculates the number of bits in an `mp_int` which is used to find the initial value of p . The call to `mp_div_2d` on line 30 calculates both the quotient q and the remainder a required. By doing both in a single function call the code size is kept fairly small. The multiplication by k is only performed if $k > 1$. This allows reductions modulo $2^p - 1$ to be performed without any multiplications.

The unsigned `s_mp_add`, `mp_cmp_mag` and `s_mp_sub` are used in place of their full sign counterparts since the inputs are only valid if they are positive. By using the unsigned versions the overhead is kept to a minimum.

Unrestricted Setup

To setup this reduction algorithm the value of $k = 2^p - n$ is required.

Algorithm **mp_reduce_2k_setup**.
Input. `mp_int` n
Output. $k = 2^p - n$

1. $p \leftarrow \lceil \lg(n) \rceil$ (`mp_count_bits`)
2. $x \leftarrow 2^p$ (`mp_2expt`)
3. $x \leftarrow x - n$ (`mp_sub`)
4. $k \leftarrow x_0$
5. Return(`MP_OKAY`).

Figure 6.18: Algorithm `mp_reduce_2k_setup`

Algorithm `mp_reduce_2k_setup`. This algorithm computes the value of

k required for the algorithm `mp_reduce_2k`. By making a temporary variable x equal to 2^p a subtraction is sufficient to solve for k . Alternatively if n has more than one digit the value of k is simply $\beta - n_0$.

```

File: bn_mp_reduce_2k_setup.c
016
017  /* determines the setup value */
018  int mp_reduce_2k_setup(mp_int *a, mp_digit *d)
019  {
020      int res, p;
021      mp_int tmp;
022
023      if ((res = mp_init(&tmp)) != MP_OKAY) {
024          return res;
025      }
026
027      p = mp_count_bits(a);
028      if ((res = mp_2expt(&tmp, p)) != MP_OKAY) {
029          mp_clear(&tmp);
030          return res;
031      }
032
033      if ((res = s_mp_sub(&tmp, a, &tmp)) != MP_OKAY) {
034          mp_clear(&tmp);
035          return res;
036      }
037
038      *d = tmp.dp[0];
039      mp_clear(&tmp);
040      return MP_OKAY;
041  }
042  #endif
043
044  /* ref:          tag: v1.0.1 */
045  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
046  /* commit time:  2017-08-29 22:27:36 +0200 */

```

Unrestricted Detection

An integer n is a valid unrestricted Diminished Radix modulus if either of the following are true.

1. The number has only one digit.

2. The number has more than one digit and every bit from the β 'th to the most significant is one.

If either condition is true then there is a power of two 2^p such that $0 < 2^p - n < \beta$. If the input is only one digit then it will always be of the correct form. Otherwise all of the bits above the first digit must be one. This arises from the fact that there will be value of k that when added to the modulus causes a carry in the first digit which propagates all the way to the most significant bit. The resulting sum will be a power of two.

Algorithm **mp_reduce_is_2k**.

Input. mp_int n

Output. 1 if of proper form, 0 otherwise

1. If $n.used = 0$ then return(0).
 2. If $n.used = 1$ then return(1).
 3. $p \leftarrow \lceil \lg(n) \rceil$ (*mp_count_bits*)
 4. for x from $\lg(\beta)$ to p do
 - 4.1 If the $(x \bmod \lg(\beta))$ 'th bit of the $\lfloor x/\lg(\beta) \rfloor$ of n is zero then return(0).
 5. Return(1).
-

Figure 6.19: Algorithm mp_reduce_is_2k

Algorithm mp_reduce_is_2k. This algorithm quickly determines if a modulus is of the form required for algorithm mp_reduce_2k to function properly.

File: bn_mp_reduce_is_2k.c

```

016
017  /* determines if mp_reduce_2k can be used */
018  int mp_reduce_is_2k(mp_int *a)
019  {
020      int ix, iy, iw;
021      mp_digit iz;
022
023      if (a->used == 0) {
024          return MP_NO;
025      } else if (a->used == 1) {
026          return MP_YES;
027      } else if (a->used > 1) {
028          iy = mp_count_bits(a);
029          iz = 1;
030          iw = 1;

```

```

031
032      /* Test every bit from the second digit up, must be 1 */
033      for (ix = DIGIT_BIT; ix < iy; ix++) {
034          if ((a->dp[iw] & iz) == 0) {
035              return MP_NO;
036          }
037          iz <= 1;
038          if (iz > (mp_digit)MP_MASK) {
039              ++iw;
040              iz = 1;
041          }
042      }
043  }
044      return MP_YES;
045  }
046
047  #endif
048
049  /* ref:          tag: v1.0.1 */
050  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
051  /* commit time: 2017-08-29 22:27:36 +0200 */

```

6.5 Algorithm Comparison

So far three very different algorithms for modular reduction have been discussed. Each of the algorithms have their own strengths and weaknesses that makes having such a selection very useful. The following table summarizes the three algorithms along with comparisons of work factors. Since all three algorithms have the restriction that $0 \leq x < n^2$ and $n > 1$ those limitations are not included in the table.

| Method | Work Required | Limitations | $m = 8$ | $m = 32$ | $m = 64$ |
|------------|----------------|-------------------|---------|----------|----------|
| Barrett | $m^2 + 2m - 1$ | None | 79 | 1087 | 4223 |
| Montgomery | $m^2 + m$ | n must be odd | 72 | 1056 | 4160 |
| D.R. | $2m$ | $n = \beta^m - k$ | 16 | 64 | 128 |

In theory Montgomery and Barrett reductions would require roughly the same amount of time to complete. However, in practice since Montgomery reduction can be written as a single function with the Comba technique it is much faster. Barrett reduction suffers from the overhead of calling the half precision multipliers, addition and division by β algorithms.

For almost every cryptographic algorithm Montgomery reduction is the algorithm of choice. The one set of algorithms where Diminished Radix reduction truly shines are based on the discrete logarithm problem such as Diffie-Hellman [12] and ElGamal [21]. In these algorithms primes of the form $\beta^m - k$ can be found and shared amongst users. These primes will allow the Diminished Radix algorithm to be used in modular exponentiation to greatly speed up the operation.

Exercises

- [3] Prove that the “trick” in algorithm `mp_montgomery_setup` actually calculates the correct value of ρ .
- [2] Devise an algorithm to reduce modulo $n + k$ for small k quickly.
- [4] Prove that the pseudo-code algorithm “Diminished Radix Reduction” (*figure 6.12*) terminates. Also prove the probability that it will terminate within $1 \leq k \leq 10$ iterations.

Chapter 7

Exponentiation

Exponentiation is the operation of raising one variable to the power of another, for example, a^b . A variant of exponentiation, computed in a finite field or ring, is called modular exponentiation. This latter style of operation is typically used in public key cryptosystems such as RSA and Diffie-Hellman. The ability to quickly compute modular exponentiations is of great benefit to any such cryptosystem and many methods have been sought to speed it up.

7.1 Exponentiation Basics

A trivial algorithm would simply multiply a against itself $b-1$ times to compute the exponentiation desired. However, as b grows in size the number of multiplications becomes prohibitive. Imagine what would happen if $b \sim 2^{1024}$ as is the case when computing an RSA signature with a 1024-bit key. Such a calculation could never be completed as it would take simply far too long.

Fortunately there is a very simple algorithm based on the laws of exponents. Recall that $\lg_a(a^b) = b$ and that $\lg_a(a^b a^c) = b + c$ which are two trivial relationships between the base and the exponent. Let b_i represent the i 'th bit of b starting from the least significant bit. If b is a k -bit integer then the following equation is true.

$$a^b = \prod_{i=0}^{k-1} a^{2^i \cdot b_i} \quad (7.1)$$

By taking the base a logarithm of both sides of the equation the following equation is the result.

$$b = \sum_{i=0}^{k-1} 2^i \cdot b_i \quad (7.2)$$

The term a^{2^i} can be found from the $i - 1$ 'th term by squaring the term since $\left(a^{2^i}\right)^2$ is equal to $a^{2^{i+1}}$. This observation forms the basis of essentially all fast exponentiation algorithms. It requires k squarings and on average $\frac{k}{2}$ multiplications to compute the result. This is indeed quite an improvement over simply multiplying by a a total of $b - 1$ times.

While this current method is a considerable speed up there are further improvements to be made. For example, the a^{2^i} term does not need to be computed in an auxiliary variable. Consider the following equivalent algorithm.

Algorithm Left to Right Exponentiation.

Input. Integer a , b and k

Output. $c = a^b$

1. $c \leftarrow 1$
 2. for i from $k - 1$ to 0 do
 - 2.1 $c \leftarrow c^2$
 - 2.2 $c \leftarrow c \cdot a^{b_i}$
 3. Return c .
-

Figure 7.1: Left to Right Exponentiation

This algorithm starts from the most significant bit and works towards the least significant bit. When the i 'th bit of b is set a is multiplied against the current product. In each iteration the product is squared which doubles the exponent of the individual terms of the product.

For example, let $b = 101100_2 \equiv 44_{10}$. The following chart demonstrates the actions of the algorithm.

| Value of i | Value of c |
|--------------|------------------------------|
| - | 1 |
| 5 | a |
| 4 | a^2 |
| 3 | $a^4 \cdot a$ |
| 2 | $a^8 \cdot a^2 \cdot a$ |
| 1 | $a^{16} \cdot a^4 \cdot a^2$ |
| 0 | $a^{32} \cdot a^8 \cdot a^4$ |

Figure 7.2: Example of Left to Right Exponentiation

When the product $a^{32} \cdot a^8 \cdot a^4$ is simplified it is equal a^{44} which is the desired exponentiation. This particular algorithm is called “Left to Right” because it reads the exponent in that order. All of the exponentiation algorithms that will be presented are of this nature.

7.1.1 Single Digit Exponentiation

The first algorithm in the series of exponentiation algorithms will be an unbounded algorithm where the exponent is a single digit. It is intended to be used when a small power of an input is required (*e.g.* a^5). It is faster than simply multiplying $b - 1$ times for all values of b that are greater than three.

Algorithm **mp_expt_d**.
Input. mp_int a and mp_digit b
Output. $c = a^b$

1. $g \leftarrow a$ (*mp_init_copy*)
2. $c \leftarrow 1$ (*mp_set*)
3. for x from 1 to $lg(\beta)$ do
 - 3.1 $c \leftarrow c^2$ (*mp_sqr*)
 - 3.2 If b AND $2^{lg(\beta)-1} \neq 0$ then
 - 3.2.1 $c \leftarrow c \cdot g$ (*mp_mul*)
 - 3.3 $b \leftarrow b \ll 1$
4. Clear g .
5. Return(*MP_OKAY*).

Figure 7.3: Algorithm mp_expt_d

Algorithm mp_expt_d. This algorithm computes the value of a raised to the power of a single digit b . It uses the left to right exponentiation algorithm to quickly compute the exponentiation. It is loosely based on algorithm 14.79 of HAC [2, pp. 615] with the difference that the exponent is a fixed width.

A copy of a is made first to allow destination variable c be the same as the source variable a . The result is set to the initial value of 1 in the subsequent step.

Inside the loop the exponent is read from the most significant bit first down to the least significant bit. First c is invariably squared on step 3.1. In the following step if the most significant bit of b is one the copy of a is multiplied against c . The value of b is shifted left one bit to make the next bit down from the most significant bit the new most significant bit. In effect each iteration of the loop moves the bits of the exponent b upwards to the most significant location.

File: bn_mp_expt_d_ex.c

```

016
017  /* calculate c = a**b using a square-multiply algorithm */
018  int mp_expt_d_ex (mp_int * a, mp_digit b, mp_int * c, int fast)
019  {
020      int      res;
021      unsigned int x;
022
023      mp_int  g;
```

```

024
025     if ((res = mp_init_copy (&g, a)) != MP_OKAY) {
026         return res;
027     }
028
029     /* set initial result */
030     mp_set (c, 1);
031
032     if (fast != 0) {
033         while (b > 0) {
034             /* if the bit is set multiply */
035             if ((b & 1) != 0) {
036                 if ((res = mp_mul (c, &g, c)) != MP_OKAY) {
037                     mp_clear (&g);
038                     return res;
039                 }
040             }
041
042             /* square */
043             if (b > 1) {
044                 if ((res = mp_sqr (&g, &g)) != MP_OKAY) {
045                     mp_clear (&g);
046                     return res;
047                 }
048             }
049
050             /* shift to next bit */
051             b >>= 1;
052         }
053     }
054     else {
055         for (x = 0; x < DIGIT_BIT; x++) {
056             /* square */
057             if ((res = mp_sqr (c, c)) != MP_OKAY) {
058                 mp_clear (&g);
059                 return res;
060             }
061
062             /* if the bit is set multiply */
063             if ((b & (mp_digit) (((mp_digit)1) << (DIGIT_BIT - 1))) != 0) {
064                 if ((res = mp_mul (c, &g, c)) != MP_OKAY) {
065                     mp_clear (&g);

```

```

066             return res;
067         }
068     }
069
070     /* shift to next bit */
071     b <<= 1;
072 }
073 } /* if ... else */
074
075     mp_clear (&g);
076     return MP_OKAY;
077 }
078 #endif
079
080 /* ref:          tag: v1.0.1 */
081 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
082 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This describes only the algorithm that is used when the parameter *fast* is 0. Line 30 sets the initial value of the result to 1. Next the loop on line 55 steps through each bit of the exponent starting from the most significant down towards the least significant. The invariant squaring operation placed on line 57 is performed first. After the squaring the result *c* is multiplied by the base *g* if and only if the most significant bit of the exponent is set. The shift on line 71 moves all of the bits of the exponent upwards towards the most significant location.

7.2 *k*-ary Exponentiation

When calculating an exponentiation the most time consuming bottleneck is the multiplications which are in general a small factor slower than squaring. Recall from the previous algorithm that b_i refers to the *i*'th bit of the exponent *b*. Suppose instead it referred to the *i*'th *k*-bit digit of the exponent of *b*. For $k = 1$ the definitions are synonymous and for $k > 1$ algorithm 7.4 computes the same exponentiation. A group of *k* bits from the exponent is called a *window*. That is it is a small window on only a portion of the entire exponent. Consider the following modification to the basic left to right exponentiation algorithm.

The squaring on step 2.1 can be calculated by squaring the value *c* successively *k* times. If the values of a^g for $0 < g < 2^k$ have been precomputed this algorithm requires only *t* multiplications and *tk* squarings. The table can be

Algorithm *k*-ary Exponentiation.

Input. Integer a , b , k and t

Output. $c = a^b$

1. $c \leftarrow 1$
 2. for i from $t - 1$ to 0 do
 - 2.1 $c \leftarrow c^{2^k}$
 - 2.2 Extract the i 'th k -bit word from b and store it in g .
 - 2.3 $c \leftarrow c \cdot a^g$
 3. Return c .
-

Figure 7.4: k -ary Exponentiation

generated with $2^{k-1} - 1$ squarings and $2^{k-1} + 1$ multiplications. This algorithm assumes that the number of bits in the exponent is evenly divisible by k . However, when it is not the remaining $0 < x \leq k - 1$ bits can be handled with algorithm 7.1.

Suppose $k = 4$ and $t = 100$. This modified algorithm will require 109 multiplications and 408 squarings to compute the exponentiation. The original algorithm would on average have required 200 multiplications and 400 squarings to compute the same value. The total number of squarings has increased slightly but the number of multiplications has nearly halved.

7.2.1 Optimal Values of k

An optimal value of k will minimize $2^k + \lceil n/k \rceil + n - 1$ for a fixed number of bits in the exponent n . The simplest approach is to brute force search amongst the values $k = 2, 3, \dots, 8$ for the lowest result. Table 7.5 lists optimal values of k for various exponent sizes and compares the number of multiplication and squarings required against algorithm 7.1.

7.2.2 Sliding-Window Exponentiation

A simple modification to the previous algorithm is only generate the upper half of the table in the range $2^{k-1} \leq g < 2^k$. Essentially this is a table for all values of g where the most significant bit of g is a one. However, in order for this to be allowed in the algorithm values of g in the range $0 \leq g < 2^{k-1}$ must be avoided.

| Exponent (bits) | Optimal k | Work at k | Work with 7.1 |
|-----------------|-------------|-------------|---------------|
| 16 | 2 | 27 | 24 |
| 32 | 3 | 49 | 48 |
| 64 | 3 | 92 | 96 |
| 128 | 4 | 175 | 192 |
| 256 | 4 | 335 | 384 |
| 512 | 5 | 645 | 768 |
| 1024 | 6 | 1257 | 1536 |
| 2048 | 6 | 2452 | 3072 |
| 4096 | 7 | 4808 | 6144 |

Figure 7.5: Optimal Values of k for k -ary Exponentiation

Table 7.6 lists optimal values of k for various exponent sizes and compares the work required against algorithm 7.4.

| Exponent (bits) | Optimal k | Work at k | Work with 7.4 |
|-----------------|-------------|-------------|---------------|
| 16 | 3 | 24 | 27 |
| 32 | 3 | 45 | 49 |
| 64 | 4 | 87 | 92 |
| 128 | 4 | 167 | 175 |
| 256 | 5 | 322 | 335 |
| 512 | 6 | 628 | 645 |
| 1024 | 6 | 1225 | 1257 |
| 2048 | 7 | 2403 | 2452 |
| 4096 | 8 | 4735 | 4808 |

Figure 7.6: Optimal Values of k for Sliding Window Exponentiation

Algorithm **Sliding Window k -ary Exponentiation**.

Input. Integer a , b , k and t

Output. $c = a^b$

1. $c \leftarrow 1$
 2. for i from $t - 1$ to 0 do
 - 2.1 If the i 'th bit of b is a zero then
 - 2.1.1 $c \leftarrow c^2$
 - 2.2 else do
 - 2.2.1 $c \leftarrow c^{2^k}$
 - 2.2.2 Extract the k bits from $(b_i b_{i-1} \dots b_{i-(k-1)})$ and store it in g .
 - 2.2.3 $c \leftarrow c \cdot a^g$
 - 2.2.4 $i \leftarrow i - k$
 3. Return c .
-

Figure 7.7: Sliding Window k -ary Exponentiation

Similar to the previous algorithm this algorithm must have a special handler when fewer than k bits are left in the exponent. While this algorithm requires the same number of squarings it can potentially have fewer multiplications. The pre-computed table a^g is also half the size as the previous table.

Consider the exponent $b = 111101011001000_2 \equiv 31432_{10}$ with $k = 3$ using both algorithms. The first algorithm will divide the exponent up as the following five 3-bit words $b \equiv (111, 101, 011, 001, 000)_2$. The second algorithm will break the exponent as $b \equiv (111, 101, 0, 110, 0, 100, 0)_2$. The single digit 0 in the second representation are where a single squaring took place instead of a squaring and multiplication. In total the first method requires 10 multiplications and 18 squarings. The second method requires 8 multiplications and 18 squarings.

In general the sliding window method is never slower than the generic k -ary method and often it is slightly faster.

7.3 Modular Exponentiation

Modular exponentiation is essentially computing the power of a base within a finite field or ring. For example, computing $d \equiv a^b \pmod{c}$ is a modular exponentiation. Instead of first computing a^b and then reducing it modulo c the intermediate result is reduced modulo c after every squaring or multiplication operation.

This guarantees that any intermediate result is bounded by $0 \leq d \leq c^2 - 2c + 1$ and can be reduced modulo c quickly using one of the algorithms presented in chapter six.

Before the actual modular exponentiation algorithm can be written a wrapper algorithm must be written first. This algorithm will allow the exponent b to be negative which is computed as $c \equiv (1/a)^{|b|} \pmod{d}$. The value of $(1/a) \pmod{c}$ is computed using the modular inverse (*see 9.4*). If no inverse exists the algorithm terminates with an error.

| |
|--|
| Algorithm mp_exptmod . |
| Input. mp_int a , b and c |
| Output. $y \equiv g^x \pmod{p}$ |

1. If $c.sign = MP_NEG$ return(MP_VAL).
2. If $b.sign = MP_NEG$ then
 - 2.1 $g' \leftarrow g^{-1} \pmod{c}$
 - 2.2 $x' \leftarrow |x|$
 - 2.3 Compute $d \equiv g'^{x'} \pmod{c}$ via recursion.
3. if p is odd **OR** p is a D.R. modulus then
 - 3.1 Compute $y \equiv g^x \pmod{p}$ via algorithm mp_exptmod_fast.
4. else
 - 4.1 Compute $y \equiv g^x \pmod{p}$ via algorithm s_mp_exptmod.

Figure 7.8: Algorithm mp_exptmod

Algorithm mp_exptmod. The first algorithm which actually performs modular exponentiation is algorithm s_mp_exptmod. It is a sliding window k -ary algorithm which uses Barrett reduction to reduce the product modulo p . The second algorithm mp_exptmod_fast performs the same operation except it uses either Montgomery or Diminished Radix reduction. The two latter reduction algorithms are clumped in the same exponentiation algorithm since their arguments are essentially the same (*two mp_ints and one mp_digit*).

File: bn_mp_exptmod.c

```

016
017
018 /* this is a shell function that calls either the normal or Montgomery
019  * exptmod functions. Originally the call to the montgomery code was
020  * embedded in the normal function but that wasted alot of stack space
021  * for nothing (since 99% of the time the Montgomery code would be called)

```

```

022  */
023  int mp_exptmod (mp_int * G, mp_int * X, mp_int * P, mp_int * Y)
024  {
025      int dr;
026
027      /* modulus P must be positive */
028      if (P->sign == MP_NEG) {
029          return MP_VAL;
030      }
031
032      /* if exponent X is negative we have to recurse */
033      if (X->sign == MP_NEG) {
034          #ifdef BN_MP_INVMOD_C
035              mp_int tmpG, tmpX;
036              int err;
037
038              /* first compute 1/G mod P */
039              if ((err = mp_init(&tmpG)) != MP_OKAY) {
040                  return err;
041              }
042              if ((err = mp_invmod(G, P, &tmpG)) != MP_OKAY) {
043                  mp_clear(&tmpG);
044                  return err;
045              }
046
047              /* now get |X| */
048              if ((err = mp_init(&tmpX)) != MP_OKAY) {
049                  mp_clear(&tmpG);
050                  return err;
051              }
052              if ((err = mp_abs(X, &tmpX)) != MP_OKAY) {
053                  mp_clear_multi(&tmpG, &tmpX, NULL);
054                  return err;
055              }
056
057              /* and now compute (1/G)**|X| instead of G**X [X < 0] */
058              err = mp_exptmod(&tmpG, &tmpX, P, Y);
059              mp_clear_multi(&tmpG, &tmpX, NULL);
060              return err;
061          #else
062              /* no invmod */
063              return MP_VAL;

```

```

064 #endif
065 }
066
067 /* modified diminished radix reduction */
068 #if defined(BN_MP_REDUCE_IS_2K_L_C) && defined(BN_MP_REDUCE_2K_L_C) && defin
    ed(BN_S_MP_EXPTMOD_C)
069     if (mp_reduce_is_2k_l(P) == MP_YES) {
070         return s_mp_exptmod(G, X, P, Y, 1);
071     }
072 #endif
073
074 #ifdef BN_MP_DR_IS_MODULUS_C
075     /* is it a DR modulus? */
076     dr = mp_dr_is_modulus(P);
077 #else
078     /* default to no */
079     dr = 0;
080 #endif
081
082 #ifdef BN_MP_REDUCE_IS_2K_C
083     /* if not, is it a unrestricted DR modulus? */
084     if (dr == 0) {
085         dr = mp_reduce_is_2k(P) << 1;
086     }
087 #endif
088
089     /* if the modulus is odd or dr != 0 use the montgomery method */
090 #ifdef BN_MP_EXPTMOD_FAST_C
091     if ((mp_isodd(P) == MP_YES) || (dr != 0)) {
092         return mp_exptmod_fast(G, X, P, Y, dr);
093     } else {
094 #endif
095 #ifdef BN_S_MP_EXPTMOD_C
096     /* otherwise use the generic Barrett reduction technique */
097     return s_mp_exptmod(G, X, P, Y, 0);
098 #else
099     /* no exptmod for evens */
100     return MP_VAL;
101 #endif
102 #ifdef BN_MP_EXPTMOD_FAST_C
103     }
104 #endif

```

```

105  }
106
107  #endif
108
109  /* ref:          tag: v1.0.1 */
110  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
111  /* commit time: 2017-08-29 22:27:36 +0200 */

```

In order to keep the algorithms in a known state the first step on line 28 is to reject any negative modulus as input. If the exponent is negative the algorithm tries to perform a modular exponentiation with the modular inverse of the base G . The temporary variable $tmpG$ is assigned the modular inverse of G and $tmpX$ is assigned the absolute value of X . The algorithm will recurse with these new values with a positive exponent.

If the exponent is positive the algorithm resumes the exponentiation. Line 76 determines if the modulus is of the restricted Diminished Radix form. If it is not line 69 attempts to determine if it is of a unrestricted Diminished Radix form. The integer dr will take on one of three values.

1. $dr = 0$ means that the modulus is not of either restricted or unrestricted Diminished Radix form.
2. $dr = 1$ means that the modulus is of restricted Diminished Radix form.
3. $dr = 2$ means that the modulus is of unrestricted Diminished Radix form.

Line 69 determines if the fast modular exponentiation algorithm can be used. It is allowed if $dr \neq 0$ or if the modulus is odd. Otherwise, the slower `s_mp_exptmod` algorithm is used which uses Barrett reduction.

7.3.1 Barrett Modular Exponentiation

Algorithm **s_mp_exptmod**.

Input. mp_int a , b and c

Output. $y \equiv g^x \pmod{p}$

1. $k \leftarrow lg(x)$
2. $win_size \leftarrow \begin{cases} 2 & \text{if } k \leq 7 \\ 3 & \text{if } 7 < k \leq 36 \\ 4 & \text{if } 36 < k \leq 140 \\ 5 & \text{if } 140 < k \leq 450 \\ 6 & \text{if } 450 < k \leq 1303 \\ 7 & \text{if } 1303 < k \leq 3529 \\ 8 & \text{if } 3529 < k \end{cases}$
3. Initialize 2^{win_size} mp_ints in an array named M and one mp_int named μ
4. Calculate the μ required for Barrett Reduction (*mp_reduce_setup*).
5. $M_1 \leftarrow g \pmod{p}$

Setup the table of small powers of g . First find $g^{2^{win_size}}$ and then all multiples of it.

6. $k \leftarrow 2^{win_size-1}$
7. $M_k \leftarrow M_1$
8. for ix from 0 to $win_size - 2$ do
 - 8.1 $M_k \leftarrow (M_k)^2 \pmod{p}$ (*mp_sqr*)
 - 8.2 $M_k \leftarrow M_k \pmod{p}$ (*mp_reduce*)
9. for ix from $2^{win_size-1} + 1$ to $2^{win_size} - 1$ do
 - 9.1 $M_{ix} \leftarrow M_{ix-1} \cdot M_1 \pmod{p}$ (*mp_mul*)
 - 9.2 $M_{ix} \leftarrow M_{ix} \pmod{p}$ (*mp_reduce*)
10. $res \leftarrow 1$

Start Sliding Window.

11. $mode \leftarrow 0, bitcnt \leftarrow 1, buf \leftarrow 0, digidx \leftarrow x.used - 1, bitcpy \leftarrow 0, bitbuf \leftarrow 0$
12. Loop
 - 12.1 $bitcnt \leftarrow bitcnt - 1$
 - 12.2 If $bitcnt = 0$ then do
 - 12.2.1 If $digidx = -1$ goto step 13.
 - 12.2.2 $buf \leftarrow x_{digidx}$
 - 12.2.3 $digidx \leftarrow digidx - 1$
 - 12.2.4 $bitcnt \leftarrow lg(\beta)$

Continued on next page.

Figure 7.9: Algorithm s_mp_exptmod

Algorithm **s_mp_exptmod** (*continued*).

Input. mp_int a , b and c

Output. $y \equiv g^x \pmod{p}$

```

12.3  $y \leftarrow (buf \gg (lg(\beta) - 1)) \text{ AND } 1$ 
12.4  $buf \leftarrow buf \ll 1$ 
12.5 if  $mode = 0$  and  $y = 0$  then goto step 12.
12.6 if  $mode = 1$  and  $y = 0$  then do
    12.6.1  $res \leftarrow res^2$ 
    12.6.2  $res \leftarrow res \pmod{p}$ 
    12.6.3 Goto step 12.
12.7  $bitcpy \leftarrow bitcpy + 1$ 
12.8  $bitbuf \leftarrow bitbuf + (y \ll (winsize - bitcpy))$ 
12.9  $mode \leftarrow 2$ 
12.10 If  $bitcpy = winsize$  then do
    Window is full so perform the squarings and single multiplication.
    12.10.1 for  $ix$  from 0 to  $winsize - 1$  do
        12.10.1.1  $res \leftarrow res^2$ 
        12.10.1.2  $res \leftarrow res \pmod{p}$ 
    12.10.2  $res \leftarrow res \cdot M_{bitbuf}$ 
    12.10.3  $res \leftarrow res \pmod{p}$ 
    Reset the window.
    12.10.4  $bitcpy \leftarrow 0, bitbuf \leftarrow 0, mode \leftarrow 1$ 

No more windows left. Check for residual bits of exponent.
13. If  $mode = 2$  and  $bitcpy > 0$  then do
    13.1 for  $ix$  from 0 to  $bitcpy - 1$  do
        13.1.1  $res \leftarrow res^2$ 
        13.1.2  $res \leftarrow res \pmod{p}$ 
        13.1.3  $bitbuf \leftarrow bitbuf \ll 1$ 
        13.1.4 If  $bitbuf \text{ AND } 2^{winsize} \neq 0$  then do
            13.1.4.1  $res \leftarrow res \cdot M_1$ 
            13.1.4.2  $res \leftarrow res \pmod{p}$ 
14.  $y \leftarrow res$ 
15. Clear  $res$ ,  $mu$  and the  $M$  array.
16. Return( $MP\_OKAY$ ).

```

Figure 7.10: Algorithm s_mp_exptmod (continued)

Algorithm s_mp_exptmod. This algorithm computes the x 'th power of g modulo p and stores the result in y . It takes advantage of the Barrett reduction

algorithm to keep the product small throughout the algorithm.

The first two steps determine the optimal window size based on the number of bits in the exponent. The larger the exponent the larger the window size becomes. After a window size *winsize* has been chosen an array of 2^{winsize} mp_int variables is allocated. This table will hold the values of $g^x \pmod p$ for $2^{\text{winsize}-1} \leq x < 2^{\text{winsize}}$.

After the table is allocated the first power of g is found. Since $g \geq p$ is allowed it must be first reduced modulo p to make the rest of the algorithm more efficient. The first element of the table at $2^{\text{winsize}-1}$ is found by squaring M_1 successively $\text{winsize} - 2$ times. The rest of the table elements are found by multiplying the previous element by M_1 modulo p .

Now that the table is available the sliding window may begin. The following list describes the functions of all the variables in the window.

1. The variable *mode* dictates how the bits of the exponent are interpreted.
 - (a) When *mode* = 0 the bits are ignored since no non-zero bit of the exponent has been seen yet. For example, if the exponent were simply 1 then there would be $\lg(\beta) - 1$ zero bits before the first non-zero bit. In this case bits are ignored until a non-zero bit is found.
 - (b) When *mode* = 1 a non-zero bit has been seen before and a new *winsize*-bit window has not been formed yet. In this mode leading 0 bits are read and a single squaring is performed. If a non-zero bit is read a new window is created.
 - (c) When *mode* = 2 the algorithm is in the middle of forming a window and new bits are appended to the window from the most significant bit downwards.
2. The variable *bitcnt* indicates how many bits are left in the current digit of the exponent left to be read. When it reaches zero a new digit is fetched from the exponent.
3. The variable *buf* holds the currently read digit of the exponent.
4. The variable *digidx* is an index into the exponents digits. It starts at the leading digit *x.used* - 1 and moves towards the trailing digit.
5. The variable *bitcpy* indicates how many bits are in the currently formed window. When it reaches *winsize* the window is flushed and the appropriate operations performed.

6. The variable *bitbuf* holds the current bits of the window being formed.

All of step 12 is the window processing loop. It will iterate while there are digits available from the exponent to read. The first step inside this loop is to extract a new digit if no more bits are available in the current digit. If there are no bits left a new digit is read and if there are no digits left then the loop terminates.

After a digit is made available step 12.3 will extract the most significant bit of the current digit and move all other bits in the digit upwards. In effect the digit is read from most significant bit to least significant bit and since the digits are read from leading to trailing edges the entire exponent is read from most significant bit to least significant bit.

At step 12.5 if the *mode* and currently extracted bit *y* are both zero the bit is ignored and the next bit is read. This prevents the algorithm from having to perform trivial squaring and reduction operations before the first non-zero bit is read. Step 12.6 and 12.7-10 handle the two cases of *mode* = 1 and *mode* = 2 respectively.



Figure 7.11: Sliding Window State Diagram

By step 13 there are no more digits left in the exponent. However, there may be partial bits in the window left. If *mode* = 2 then a Left-to-Right algorithm is used to process the remaining few bits.

File: bn_s_mp_exptmod.c

```

016 #ifdef MP_LOW_MEM
017     #define TAB_SIZE 32
018 #else
019     #define TAB_SIZE 256
020 #endif
021
022 int s_mp_exptmod (mp_int * G, mp_int * X, mp_int * P, mp_int * Y, int redmod
023 e)
024 {
025     mp_int M[TAB_SIZE], res, mu;
026     mp_digit buf;
027     int err, bitbuf, bitcpy, bitcnt, mode, digidx, x, y, winsize;
028     int (*redux)(mp_int*,mp_int*,mp_int*);
029
030     /* find window size */
031     x = mp_count_bits (X);
032     if (x <= 7) {
033         winsize = 2;
034     } else if (x <= 36) {
035         winsize = 3;
036     } else if (x <= 140) {
037         winsize = 4;
038     } else if (x <= 450) {
039         winsize = 5;
040     } else if (x <= 1303) {
041         winsize = 6;
042     } else if (x <= 3529) {
043         winsize = 7;
044     } else {
045         winsize = 8;
046     }
047
048 #ifdef MP_LOW_MEM
049     if (winsize > 5) {
050         winsize = 5;
051     }
052 #endif
053
054     /* init M array */
055     /* init first cell */
056     if ((err = mp_init(&M[1])) != MP_OKAY) {
057         return err;
058     }

```

```

057     }
058
059     /* now init the second half of the array */
060     for (x = 1<<(winsize-1); x < (1 << winsize); x++) {
061         if ((err = mp_init(&M[x])) != MP_OKAY) {
062             for (y = 1<<(winsize-1); y < x; y++) {
063                 mp_clear (&M[y]);
064             }
065             mp_clear(&M[1]);
066             return err;
067         }
068     }
069
070     /* create mu, used for Barrett reduction */
071     if ((err = mp_init (&mu)) != MP_OKAY) {
072         goto LBL_M;
073     }
074
075     if (redmode == 0) {
076         if ((err = mp_reduce_setup (&mu, P)) != MP_OKAY) {
077             goto LBL_MU;
078         }
079         redux = mp_reduce;
080     } else {
081         if ((err = mp_reduce_2k_setup_1 (P, &mu)) != MP_OKAY) {
082             goto LBL_MU;
083         }
084         redux = mp_reduce_2k_1;
085     }
086
087     /* create M table
088     *
089     * The M table contains powers of the base,
090     * e.g. M[x] = G**x mod P
091     *
092     * The first half of the table is not
093     * computed though accept for M[0] and M[1]
094     */
095     if ((err = mp_mod (G, P, &M[1])) != MP_OKAY) {
096         goto LBL_MU;
097     }
098

```

```

099  /* compute the value at M[1<<(winsize-1)] by squaring
100  * M[1] (winsize-1) times
101  */
102  if ((err = mp_copy (&M[1], &M[1 << (winsize - 1)])) != MP_OKAY) {
103      goto LBL_MU;
104  }
105
106  for (x = 0; x < (winsize - 1); x++) {
107      /* square it */
108      if ((err = mp_sqr (&M[1 << (winsize - 1)],
109                      &M[1 << (winsize - 1)])) != MP_OKAY) {
110          goto LBL_MU;
111      }
112
113      /* reduce modulo P */
114      if ((err = redux (&M[1 << (winsize - 1)], P, &mu)) != MP_OKAY) {
115          goto LBL_MU;
116      }
117  }
118
119  /* create upper table, that is M[x] = M[x-1] * M[1] (mod P)
120  * for x = (2**(winsize - 1) + 1) to (2**winsize - 1)
121  */
122  for (x = (1 << (winsize - 1)) + 1; x < (1 << winsize); x++) {
123      if ((err = mp_mul (&M[x - 1], &M[1], &M[x])) != MP_OKAY) {
124          goto LBL_MU;
125      }
126      if ((err = redux (&M[x], P, &mu)) != MP_OKAY) {
127          goto LBL_MU;
128      }
129  }
130
131  /* setup result */
132  if ((err = mp_init (&res)) != MP_OKAY) {
133      goto LBL_MU;
134  }
135  mp_set (&res, 1);
136
137  /* set initial mode and bit cnt */
138  mode = 0;
139  bitcnt = 1;
140  buf = 0;

```

```

141     digidx = X->used - 1;
142     bitcpy = 0;
143     bitbuf = 0;
144
145     for (;;) {
146         /* grab next digit as required */
147         if (--bitcnt == 0) {
148             /* if digidx == -1 we are out of digits */
149             if (digidx == -1) {
150                 break;
151             }
152             /* read next digit and reset the bitcnt */
153             buf = X->dp[digidx--];
154             bitcnt = (int) DIGIT_BIT;
155         }
156
157         /* grab the next msb from the exponent */
158         y = (buf >> (mp_digit)(DIGIT_BIT - 1)) & 1;
159         buf <<= (mp_digit)1;
160
161         /* if the bit is zero and mode == 0 then we ignore it
162          * These represent the leading zero bits before the first 1 bit
163          * in the exponent. Technically this opt is not required but it
164          * does lower the # of trivial squaring/reductions used
165          */
166         if ((mode == 0) && (y == 0)) {
167             continue;
168         }
169
170         /* if the bit is zero and mode == 1 then we square */
171         if ((mode == 1) && (y == 0)) {
172             if ((err = mp_sqr (&res, &res)) != MP_OKAY) {
173                 goto LBL_RES;
174             }
175             if ((err = redux (&res, P, &mu)) != MP_OKAY) {
176                 goto LBL_RES;
177             }
178             continue;
179         }
180
181         /* else we add it to the window */
182         bitbuf |= (y << (winsize - ++bitcpy));

```

```

183     mode    = 2;
184
185     if (bitcpy == winsize) {
186         /* ok window is filled so square as required and multiply */
187         /* square first */
188         for (x = 0; x < winsize; x++) {
189             if ((err = mp_sqr (&res, &res)) != MP_OKAY) {
190                 goto LBL_RES;
191             }
192             if ((err = redux (&res, P, &mu)) != MP_OKAY) {
193                 goto LBL_RES;
194             }
195         }
196
197         /* then multiply */
198         if ((err = mp_mul (&res, &M[bitbuf], &res)) != MP_OKAY) {
199             goto LBL_RES;
200         }
201         if ((err = redux (&res, P, &mu)) != MP_OKAY) {
202             goto LBL_RES;
203         }
204
205         /* empty window and reset */
206         bitcpy = 0;
207         bitbuf = 0;
208         mode    = 1;
209     }
210 }
211
212 /* if bits remain then square/multiply */
213 if ((mode == 2) && (bitcpy > 0)) {
214     /* square then multiply if the bit is set */
215     for (x = 0; x < bitcpy; x++) {
216         if ((err = mp_sqr (&res, &res)) != MP_OKAY) {
217             goto LBL_RES;
218         }
219         if ((err = redux (&res, P, &mu)) != MP_OKAY) {
220             goto LBL_RES;
221         }
222
223         bitbuf <<= 1;
224         if ((bitbuf & (1 << winsize)) != 0) {

```

```

225         /* then multiply */
226         if ((err = mp_mul (&res, &M[1], &res)) != MP_OKAY) {
227             goto LBL_RES;
228         }
229         if ((err = redux (&res, P, &mu)) != MP_OKAY) {
230             goto LBL_RES;
231         }
232     }
233 }
234 }
235
236 mp_exch (&res, Y);
237 err = MP_OKAY;
238 LBL_RES: mp_clear (&res);
239 LBL_MU: mp_clear (&mu);
240 LBL_M:
241     mp_clear (&M[1]);
242     for (x = 1 << (winsize-1); x < (1 << winsize); x++) {
243         mp_clear (&M[x]);
244     }
245     return err;
246 }
247 #endif
248
249 /* ref:          tag: v1.0.1 */
250 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
251 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Lines 31 through 45 determine the optimal window size based on the length of the exponent in bits. The window divisions are sorted from smallest to greatest so that in each **if** statement only one condition must be tested. For example, by the **if** statement on line 37 the value of x is already known to be greater than 140.

The conditional piece of code beginning on line 47 allows the window size to be restricted to five bits. This logic is used to ensure the table of precomputed powers of G remains relatively small.

The for loop on line 60 initializes the M array while lines 71 and 76 through 85 initialize the reduction function that will be used for this modulus.

– More later.

7.4 Quick Power of Two

Calculating $b = 2^a$ can be performed much quicker than with any of the previous algorithms. Recall that a logical shift left $m \ll k$ is equivalent to $m \cdot 2^k$. By this logic when $m = 1$ a quick power of two can be achieved.

Algorithm **mp_2expt**.

Input. integer b

Output. $a \leftarrow 2^b$

1. $a \leftarrow 0$
 2. If $a.alloc < \lfloor b/lg(\beta) \rfloor + 1$ then grow a appropriately.
 3. $a.used \leftarrow \lfloor b/lg(\beta) \rfloor + 1$
 4. $a_{\lfloor b/lg(\beta) \rfloor} \leftarrow 1 \ll (b \bmod lg(\beta))$
 5. Return(*MP_OKAY*).
-

Figure 7.12: Algorithm mp_2expt

Algorithm mp_2expt.

File: bn_mp_2expt.c

```

016
017  /* computes a = 2**b
018  *
019  * Simple algorithm which zeroes the int, grows it then just sets one bit
020  * as required.
021  */
022  int
023  mp_2expt (mp_int * a, int b)
024  {
025      int      res;
026
027      /* zero a as per default */
028      mp_zero (a);
029
030      /* grow a to accomodate the single bit */
031      if ((res = mp_grow (a, (b / DIGIT_BIT) + 1)) != MP_OKAY) {
032          return res;
033      }
034
035      /* set the used count of where the bit will go */

```



```
036     a->used = (b / DIGIT_BIT) + 1;
037
038     /* put the single bit in its place */
039     a->dp[b / DIGIT_BIT] = ((mp_digit)1) << (b % DIGIT_BIT);
040
041     return MP_OKAY;
042 }
043 #endif
044
045 /* ref:          tag: v1.0.1 */
046 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
047 /* commit time: 2017-08-29 22:27:36 +0200 */
```


Chapter 8

Higher Level Algorithms

This chapter discusses the various higher level algorithms that are required to complete a well rounded multiple precision integer package. These routines are less performance oriented than the algorithms of chapters five, six and seven but are no less important.

The first section describes a method of integer division with remainder that is universally well known. It provides the signed division logic for the package. The subsequent section discusses a set of algorithms which allow a single digit to be the 2nd operand for a variety of operations. These algorithms serve mostly to simplify other algorithms where small constants are required. The last two sections discuss how to manipulate various representations of integers. For example, converting from an `mp_int` to a string of character.

8.1 Integer Division with Remainder

Integer division aside from modular exponentiation is the most intensive algorithm to compute. Like addition, subtraction and multiplication the basis of this algorithm is the long-hand division algorithm taught to school children. Throughout this discussion several common variables will be used. Let x represent the divisor and y represent the dividend. Let q represent the integer quotient $\lfloor y/x \rfloor$ and let r represent the remainder $r = y - x\lfloor y/x \rfloor$. The following simple algorithm will be used to start the discussion.

Algorithm **Radix- β Integer Division.**

Input. integer x and y

Output. $q = \lfloor y/x \rfloor, r = y - xq$

1. $q \leftarrow 0$
 2. $n \leftarrow ||y|| - ||x||$
 3. for t from n down to 0 do
 - 3.1 Maximize k such that $kx\beta^t$ is less than or equal to y and $(k+1)x\beta^t$ is greater.
 - 3.2 $q \leftarrow q + k\beta^t$
 - 3.3 $y \leftarrow y - kx\beta^t$
 4. $r \leftarrow y$
 5. Return(q, r)
-

Figure 8.1: Algorithm Radix- β Integer Division

As children we are taught this very simple algorithm for the case of $\beta = 10$. Almost instinctively several optimizations are taught for which their reason of existing are never explained. For this example let $y = 5471$ represent the dividend and $x = 23$ represent the divisor.

To find the first digit of the quotient the value of k must be maximized such that $kx\beta^t$ is less than or equal to y and simultaneously $(k+1)x\beta^t$ is greater than y . Implicitly k is the maximum value the t 'th digit of the quotient may have. The habitual method used to find the maximum is to "eyeball" the two numbers, typically only the leading digits and quickly estimate a quotient. By only using leading digits a much simpler division may be used to form an educated guess at what the value must be. In this case $k = \lfloor 54/23 \rfloor = 2$ quickly arises as a possible solution. Indeed $2x\beta^2 = 4600$ is less than $y = 5471$ and simultaneously $(k+1)x\beta^2 = 6900$ is larger than y . As a result $k\beta^2$ is added to the quotient which now equals $q = 200$ and 4600 is subtracted from y to give a remainder of $y = 841$.

Again this process is repeated to produce the quotient digit $k = 3$ which makes the quotient $q = 200 + 3\beta = 230$ and the remainder $y = 841 - 3x\beta = 181$. Finally the last iteration of the loop produces $k = 7$ which leads to the quotient $q = 230 + 7 = 237$ and the remainder $y = 181 - 7x = 20$. The final quotient and remainder found are $q = 237$ and $r = y = 20$ which are indeed correct since $237 \cdot 23 + 20 = 5471$ is true.

8.1.1 Quotient Estimation

As alluded to earlier the quotient digit k can be estimated from only the leading digits of both the divisor and dividend. When p leading digits are used from both the divisor and dividend to form an estimation the accuracy of the estimation rises as p grows. Technically speaking the estimation is based on assuming the lower $\|y\| - p$ and $\|x\| - p$ lower digits of the dividend and divisor are zero.

The value of the estimation may off by a few values in either direction and in general is fairly correct. A simplification [1, pp. 271] of the estimation technique is to use $t + 1$ digits of the dividend and t digits of the divisor, in particularly when $t = 1$. The estimate using this technique is never too small. For the following proof let $t = \|y\| - 1$ and $s = \|x\| - 1$ represent the most significant digits of the dividend and divisor respectively.

Proof. *The quotient $\hat{k} = \lfloor (y_t\beta + y_{t-1})/x_s \rfloor$ is greater than or equal to $k = \lfloor y/(x \cdot \beta^{\|y\| - \|x\| - 1}) \rfloor$.* The first obvious case is when $\hat{k} = \beta - 1$ in which case the proof is concluded since the real quotient cannot be larger. For all other cases $\hat{k} = \lfloor (y_t\beta + y_{t-1})/x_s \rfloor$ and $\hat{k}x_s \geq y_t\beta + y_{t-1} - x_s + 1$. The latter portion of the inequality $-x_s + 1$ arises from the fact that a truncated integer division will give the same quotient for at most $x_s - 1$ values. Next a series of inequalities will prove the hypothesis.

$$y - \hat{k}x \leq y - \hat{k}x_s\beta^s \quad (8.1)$$

This is trivially true since $x \geq x_s\beta^s$. Next we replace $\hat{k}x_s\beta^s$ by the previous inequality for $\hat{k}x_s$.

$$y - \hat{k}x \leq y_t\beta^t + \dots + y_0 - (y_t\beta^t + y_{t-1}\beta^{t-1} - x_s\beta^t + \beta^s) \quad (8.2)$$

By simplifying the previous inequality the following inequality is formed.

$$y - \hat{k}x \leq y_{t-2}\beta^{t-2} + \dots + y_0 + x_s\beta^s - \beta^s \quad (8.3)$$

Subsequently,

$$y_{t-2}\beta^{t-2} + \dots + y_0 + x_s\beta^s - \beta^s < x_s\beta^s \leq x \quad (8.4)$$

Which proves that $y - \hat{k}x \leq x$ and by consequence $\hat{k} \geq k$ which concludes the proof. **QED**

8.1.2 Normalized Integers

For the purposes of division a normalized input is when the divisors leading digit x_n is greater than or equal to $\beta/2$. By multiplying both x and y by $j = \lfloor (\beta/2)/x_n \rfloor$ the quotient remains unchanged and the remainder is simply j times the original remainder. The purpose of normalization is to ensure the leading digit of the divisor is sufficiently large such that the estimated quotient will lie in the domain of a single digit. Consider the maximum dividend $(\beta-1) \cdot \beta + (\beta-1)$ and the minimum divisor $\beta/2$.

$$\frac{\beta^2 - 1}{\beta/2} \leq 2\beta - \frac{2}{\beta} \quad (8.5)$$

At most the quotient approaches 2β , however, in practice this will not occur since that would imply the previous quotient digit was too small.

8.1.3 Radix- β Division with Remainder

Algorithm **mp_div**.

Input. mp_int a, b

Output. $c = \lfloor a/b \rfloor$, $d = a - bc$

1. If $b = 0$ return(MP_VAL).
2. If $|a| < |b|$ then do
 - 2.1 $d \leftarrow a$
 - 2.2 $c \leftarrow 0$
 - 2.3 Return(MP_OKAY).

Setup the quotient to receive the digits.

3. Grow q to $a.used + 2$ digits.
4. $q \leftarrow 0$
5. $x \leftarrow |a|, y \leftarrow |b|$
6. $sign \leftarrow \begin{cases} MP_ZPOS & \text{if } a.sign = b.sign \\ MP_NEG & \text{otherwise} \end{cases}$

Normalize the inputs such that the leading digit of y is greater than or equal to $\beta/2$.

7. $norm \leftarrow (lg(\beta) - 1) - (\lceil lg(y) \rceil \pmod{lg(\beta)})$
8. $x \leftarrow x \cdot 2^{norm}, y \leftarrow y \cdot 2^{norm}$

Find the leading digit of the quotient.

9. $n \leftarrow x.used - 1, t \leftarrow y.used - 1$
10. $y \leftarrow y \cdot \beta^{n-t}$
11. While $(x \geq y)$ do
 - 11.1 $q_{n-t} \leftarrow q_{n-t} + 1$
 - 11.2 $x \leftarrow x - y$
12. $y \leftarrow \lfloor y / \beta^{n-t} \rfloor$

Continued on the next page.

Figure 8.2: Algorithm mp_div

Algorithm **mp_div** (continued).

Input. mp_int a, b

Output. $c = \lfloor a/b \rfloor, d = a - bc$

Now find the remainder for the digits.

13. for i from n down to $(t + 1)$ do
 - 13.1 If $i > x.used$ then jump to the next iteration of this loop.
 - 13.2 If $x_i = y_t$ then
 - 13.2.1 $q_{i-t-1} \leftarrow \beta - 1$
 - 13.3 else
 - 13.3.1 $\hat{r} \leftarrow x_i \cdot \beta + x_{i-1}$
 - 13.3.2 $\hat{r} \leftarrow \lfloor \hat{r}/y_t \rfloor$
 - 13.3.3 $q_{i-t-1} \leftarrow \hat{r}$
 - 13.4 $q_{i-t-1} \leftarrow q_{i-t-1} + 1$

Fixup quotient estimation.

- 13.5 Loop
 - 13.5.1 $q_{i-t-1} \leftarrow q_{i-t-1} - 1$
 - 13.5.2 $t1 \leftarrow 0$
 - 13.5.3 $t1_0 \leftarrow y_{t-1}, t1_1 \leftarrow y_t, t1.used \leftarrow 2$
 - 13.5.4 $t1 \leftarrow t1 \cdot q_{i-t-1}$
 - 13.5.5 $t2_0 \leftarrow x_{i-2}, t2_1 \leftarrow x_{i-1}, t2_2 \leftarrow x_i, t2.used \leftarrow 3$
 - 13.5.6 If $|t1| > |t2|$ then goto step 13.5.
 - 13.6 $t1 \leftarrow y \cdot q_{i-t-1}$
 - 13.7 $t1 \leftarrow t1 \cdot \beta^{i-t-1}$
 - 13.8 $x \leftarrow x - t1$
 - 13.9 If $x.sign = MP_NEG$ then
 - 13.10 $t1 \leftarrow y$
 - 13.11 $t1 \leftarrow t1 \cdot \beta^{i-t-1}$
 - 13.12 $x \leftarrow x + t1$
 - 13.13 $q_{i-t-1} \leftarrow q_{i-t-1} - 1$

Finalize the result.

14. Clamp excess digits of q
 15. $c \leftarrow q, c.sign \leftarrow sign$
 16. $x.sign \leftarrow a.sign$
 17. $d \leftarrow \lfloor x/2^{norm} \rfloor$
 18. Return(MP_OKAY).
-

Figure 8.3: Algorithm mp_div (continued)

Algorithm mp_div. This algorithm will calculate quotient and remainder from an integer division given a dividend and divisor. The algorithm is a signed division and will produce a fully qualified quotient and remainder.

First the divisor b must be non-zero which is enforced in step one. If the divisor is larger than the dividend then the quotient is implicitly zero and the remainder is the dividend.

After the first two trivial cases of inputs are handled the variable q is setup to receive the digits of the quotient. Two unsigned copies of the divisor y and dividend x are made as well. The core of the division algorithm is an unsigned division and will only work if the values are positive. Now the two values x and y must be normalized such that the leading digit of y is greater than or equal to $\beta/2$. This is performed by shifting both to the left by enough bits to get the desired normalization.

At this point the division algorithm can begin producing digits of the quotient. Recall that maximum value of the estimation used is $2\beta - \frac{2}{\beta}$ which means that a digit of the quotient must be first produced by another means. In this case y is shifted to the left (*step ten*) so that it has the same number of digits as x . The loop on step eleven will subtract multiples of the shifted copy of y until x is smaller. Since the leading digit of y is greater than or equal to $\beta/2$ this loop will iterate at most two times to produce the desired leading digit of the quotient.

Now the remainder of the digits can be produced. The equation $\hat{q} = \lfloor \frac{x_i\beta + x_{i-1}}{y_t} \rfloor$ is used to fairly accurately approximate the true quotient digit. The estimation can in theory produce an estimation as high as $2\beta - \frac{2}{\beta}$ but by induction the upper quotient digit is correct (*as established on step eleven*) and the estimate must be less than β .

Recall from section 8.1.1 that the estimation is never too low but may be too high. The next step of the estimation process is to refine the estimation. The loop on step 13.5 uses $x_i\beta^2 + x_{i-1}\beta + x_{i-2}$ and $q_{i-t-1}(y_t\beta + y_{t-1})$ as a higher order approximation to adjust the quotient digit.

After both phases of estimation the quotient digit may still be off by a value of one¹. Steps 13.6 and 13.7 subtract the multiple of the divisor from the dividend (*Similar to step 3.3 of algorithm 8.1*) and then subsequently add a multiple of the divisor if the quotient was too large.

Now that the quotient has been determined finalizing the result is a matter of clamping the quotient, fixing the sizes and de-normalizing the remainder. An important aspect of this algorithm seemingly overlooked in other descriptions

¹This is similar to the error introduced by optimizing Barrett reduction.

such as that of Algorithm 14.20 HAC [2, pp. 598] is that when the estimations are being made (*inside the loop on step 13.5*) that the digits y_{t-1} , x_{i-2} and x_{i-1} may lie outside their respective boundaries. For example, if $t = 0$ or $i \leq 1$ then the digits would be undefined. In those cases the digits should respectively be replaced with a zero.

File: bn_mp_div.c

```

016
017 #ifdef BN_MP_DIV_SMALL
018
019 /* slower bit-bang division... also smaller */
020 int mp_div(mp_int * a, mp_int * b, mp_int * c, mp_int * d)
021 {
022     mp_int ta, tb, tq, q;
023     int     res, n, n2;
024
025     /* is divisor zero ? */
026     if (mp_iszero (b) == MP_YES) {
027         return MP_VAL;
028     }
029
030     /* if a < b then q=0, r = a */
031     if (mp_cmp_mag (a, b) == MP_LT) {
032         if (d != NULL) {
033             res = mp_copy (a, d);
034         } else {
035             res = MP_OKAY;
036         }
037         if (c != NULL) {
038             mp_zero (c);
039         }
040         return res;
041     }
042
043     /* init our temps */
044     if ((res = mp_init_multi(&ta, &tb, &tq, &q, NULL)) != MP_OKAY) {
045         return res;
046     }
047
048
049     mp_set(&tq, 1);
050     n = mp_count_bits(a) - mp_count_bits(b);

```

```

051     if ((res = mp_abs(a, &ta)) != MP_OKAY) ||
052         ((res = mp_abs(b, &tb)) != MP_OKAY) ||
053         ((res = mp_mul_2d(&tb, n, &tb)) != MP_OKAY) ||
054         ((res = mp_mul_2d(&tq, n, &tq)) != MP_OKAY)) {
055         goto LBL_ERR;
056     }
057
058     while (n-- >= 0) {
059         if (mp_cmp(&tb, &ta) != MP_GT) {
060             if ((res = mp_sub(&ta, &tb, &ta)) != MP_OKAY) ||
061                 ((res = mp_add(&q, &tq, &q)) != MP_OKAY)) {
062                 goto LBL_ERR;
063             }
064         }
065         if ((res = mp_div_2d(&tb, 1, &tb, NULL)) != MP_OKAY) ||
066             ((res = mp_div_2d(&tq, 1, &tq, NULL)) != MP_OKAY)) {
067             goto LBL_ERR;
068         }
069     }
070
071     /* now q == quotient and ta == remainder */
072     n = a->sign;
073     n2 = (a->sign == b->sign) ? MP_ZPOS : MP_NEG;
074     if (c != NULL) {
075         mp_exch(c, &q);
076         c->sign = (mp_iszero(c) == MP_YES) ? MP_ZPOS : n2;
077     }
078     if (d != NULL) {
079         mp_exch(d, &ta);
080         d->sign = (mp_iszero(d) == MP_YES) ? MP_ZPOS : n;
081     }
082 LBL_ERR:
083     mp_clear_multi(&ta, &tb, &tq, &q, NULL);
084     return res;
085 }
086
087 #else
088
089 /* integer signed division.
090  * c*b + d == a [e.g. a/b, c=quotient, d=remainder]
091  * HAC pp.598 Algorithm 14.20
092  */

```

```

093  * Note that the description in HAC is horribly
094  * incomplete. For example, it doesn't consider
095  * the case where digits are removed from 'x' in
096  * the inner loop. It also doesn't consider the
097  * case that y has fewer than three digits, etc..
098  *
099  * The overall algorithm is as described as
100  * 14.20 from HAC but fixed to treat these cases.
101  */
102  int mp_div (mp_int * a, mp_int * b, mp_int * c, mp_int * d)
103  {
104      mp_int  q, x, y, t1, t2;
105      int      res, n, t, i, norm, neg;
106
107      /* is divisor zero ? */
108      if (mp_iszero (b) == MP_YES) {
109          return MP_VAL;
110      }
111
112      /* if a < b then q=0, r = a */
113      if (mp_cmp_mag (a, b) == MP_LT) {
114          if (d != NULL) {
115              res = mp_copy (a, d);
116          } else {
117              res = MP_OKAY;
118          }
119          if (c != NULL) {
120              mp_zero (c);
121          }
122          return res;
123      }
124
125      if ((res = mp_init_size (&q, a->used + 2)) != MP_OKAY) {
126          return res;
127      }
128      q.used = a->used + 2;
129
130      if ((res = mp_init (&t1)) != MP_OKAY) {
131          goto LBL_Q;
132      }
133
134      if ((res = mp_init (&t2)) != MP_OKAY) {

```

```

135     goto LBL_T1;
136 }
137
138 if ((res = mp_init_copy (&x, a)) != MP_OKAY) {
139     goto LBL_T2;
140 }
141
142 if ((res = mp_init_copy (&y, b)) != MP_OKAY) {
143     goto LBL_X;
144 }
145
146 /* fix the sign */
147 neg = (a->sign == b->sign) ? MP_ZPOS : MP_NEG;
148 x.sign = y.sign = MP_ZPOS;
149
150 /* normalize both x and y, ensure that y >= b/2, [b == 2**DIGIT_BIT] */
151 norm = mp_count_bits(&y) % DIGIT_BIT;
152 if (norm < (int)(DIGIT_BIT-1)) {
153     norm = (DIGIT_BIT-1) - norm;
154     if ((res = mp_mul_2d (&x, norm, &x)) != MP_OKAY) {
155         goto LBL_Y;
156     }
157     if ((res = mp_mul_2d (&y, norm, &y)) != MP_OKAY) {
158         goto LBL_Y;
159     }
160 } else {
161     norm = 0;
162 }
163
164 /* note hac does 0 based, so if used==5 then its 0,1,2,3,4, e.g. use 4 */
165 n = x.used - 1;
166 t = y.used - 1;
167
168 /* while (x >= y*b**n-t) do { q[n-t] += 1; x -= y*b**{n-t} } */
169 if ((res = mp_lshd (&y, n - t)) != MP_OKAY) { /* y = y*b**{n-t} */
170     goto LBL_Y;
171 }
172
173 while (mp_cmp (&x, &y) != MP_LT) {
174     ++(q.dp[n - t]);
175     if ((res = mp_sub (&x, &y, &x)) != MP_OKAY) {
176         goto LBL_Y;

```

```

177     }
178 }
179
180 /* reset y by shifting it back down */
181 mp_rshd (&y, n - t);
182
183 /* step 3. for i from n down to (t + 1) */
184 for (i = n; i >= (t + 1); i--) {
185     if (i > x.used) {
186         continue;
187     }
188
189     /* step 3.1 if xi == yt then set q{i-t-1} to b-1,
190      * otherwise set q{i-t-1} to (xi*b + x{i-1})/yt */
191     if (x.dp[i] == y.dp[t]) {
192         q.dp[(i - t) - 1] = (((mp_digit)1) << DIGIT_BIT) - 1;
193     } else {
194         mp_word tmp;
195         tmp = ((mp_word) x.dp[i]) << ((mp_word) DIGIT_BIT);
196         tmp |= ((mp_word) x.dp[i - 1]);
197         tmp /= ((mp_word) y.dp[t]);
198         if (tmp > (mp_word) MP_MASK) {
199             tmp = MP_MASK;
200         }
201         q.dp[(i - t) - 1] = (mp_digit) (tmp & (mp_word) (MP_MASK));
202     }
203
204     /* while (q{i-t-1} * (yt * b + y{t-1})) >
205        xi * b**2 + xi-1 * b + xi-2
206
207        do q{i-t-1} -= 1;
208 */
209     q.dp[(i - t) - 1] = (q.dp[(i - t) - 1] + 1) & MP_MASK;
210     do {
211         q.dp[(i - t) - 1] = (q.dp[(i - t) - 1] - 1) & MP_MASK;
212
213         /* find left hand */
214         mp_zero (&t1);
215         t1.dp[0] = ((t - 1) < 0) ? 0 : y.dp[t - 1];
216         t1.dp[1] = y.dp[t];
217         t1.used = 2;
218         if ((res = mp_mul_d (&t1, q.dp[(i - t) - 1], &t1)) != MP_OKAY) {

```

```

219         goto LBL_Y;
220     }
221
222     /* find right hand */
223     t2.dp[0] = ((i - 2) < 0) ? 0 : x.dp[i - 2];
224     t2.dp[1] = ((i - 1) < 0) ? 0 : x.dp[i - 1];
225     t2.dp[2] = x.dp[i];
226     t2.used = 3;
227     } while (mp_cmp_mag(&t1, &t2) == MP_GT);
228
229     /* step 3.3 x = x - q{i-t-1} * y * b**{i-t-1} */
230     if ((res = mp_mul_d (&y, q.dp[(i - t) - 1], &t1)) != MP_OKAY) {
231         goto LBL_Y;
232     }
233
234     if ((res = mp_lshd (&t1, (i - t) - 1)) != MP_OKAY) {
235         goto LBL_Y;
236     }
237
238     if ((res = mp_sub (&x, &t1, &x)) != MP_OKAY) {
239         goto LBL_Y;
240     }
241
242     /* if x < 0 then { x = x + y*b**{i-t-1}; q{i-t-1} -= 1; } */
243     if (x.sign == MP_NEG) {
244         if ((res = mp_copy (&y, &t1)) != MP_OKAY) {
245             goto LBL_Y;
246         }
247         if ((res = mp_lshd (&t1, (i - t) - 1)) != MP_OKAY) {
248             goto LBL_Y;
249         }
250         if ((res = mp_add (&x, &t1, &x)) != MP_OKAY) {
251             goto LBL_Y;
252         }
253     }
254     q.dp[(i - t) - 1] = (q.dp[(i - t) - 1] - 1UL) & MP_MASK;
255 }
256 }
257
258 /* now q is the quotient and x is the remainder
259  * [which we have to normalize]
260  */

```

```

261
262     /* get sign before writing to c */
263     x.sign = (x.used == 0) ? MP_ZPOS : a->sign;
264
265     if (c != NULL) {
266         mp_clamp (&q);
267         mp_exch (&q, c);
268         c->sign = neg;
269     }
270
271     if (d != NULL) {
272         if ((res = mp_div_2d (&x, norm, &x, NULL)) != MP_OKAY) {
273             goto LBL_Y;
274         }
275         mp_exch (&x, d);
276     }
277
278     res = MP_OKAY;
279
280     LBL_Y:mp_clear (&y);
281     LBL_X:mp_clear (&x);
282     LBL_T2:mp_clear (&t2);
283     LBL_T1:mp_clear (&t1);
284     LBL_Q:mp_clear (&q);
285     return res;
286 }
287
288 #endif
289
290 #endif
291
292     /* ref:          tag: v1.0.1 */
293     /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
294     /* commit time:  2017-08-29 22:27:36 +0200 */

```

The implementation of this algorithm differs slightly from the pseudo code presented previously. In this algorithm either of the quotient c or remainder d may be passed as a **NULL** pointer which indicates their value is not desired. For example, the C code to call the division algorithm with only the quotient is

```
mp_div(&a, &b, &c, NULL); /* c = [a/b] */
```

Lines 108 and 113 handle the two trivial cases of inputs which are division

by zero and dividend smaller than the divisor respectively. After the two trivial cases all of the temporary variables are initialized. Line 147 determines the sign of the quotient and line 148 ensures that both x and y are positive.

The number of bits in the leading digit is calculated on line 151. Implicitly an `mp_int` with r digits will require $lg(\beta)(r-1) + k$ bits of precision which when reduced modulo $lg(\beta)$ produces the value of k . In this case k is the number of bits in the leading digit which is exactly what is required. For the algorithm to operate k must equal $lg(\beta) - 1$ and when it does not the inputs must be normalized by shifting them to the left by $lg(\beta) - 1 - k$ bits.

Throughout the variables n and t will represent the highest digit of x and y respectively. These are first used to produce the leading digit of the quotient. The loop beginning on line 184 will produce the remainder of the quotient digits.

The conditional “continue” on line 186 is used to prevent the algorithm from reading past the leading edge of x which can occur when the algorithm eliminates multiple non-zero digits in a single iteration. This ensures that x_i is always non-zero since by definition the digits above the i 'th position x must be zero in order for the quotient to be precise².

Lines 214, 216 and 223 through 225 manually construct the high accuracy estimations by setting the digits of the two `mp_int` variables directly.

8.2 Single Digit Helpers

This section briefly describes a series of single digit helper algorithms which come in handy when working with small constants. All of the helper functions assume the single digit input is positive and will treat them as such.

8.2.1 Single Digit Addition and Subtraction

Both addition and subtraction are performed by “cheating” and using `mp_set` followed by the higher level addition or subtraction algorithms. As a result these algorithms are substantially simpler with a slight cost in performance.

²Precise as far as integer division is concerned.

Algorithm **mp_add_d**.
Input. mp_int a and a mp_digit b
Output. $c = a + b$

1. $t \leftarrow b$ (*mp_set*)
2. $c \leftarrow a + t$
3. Return(*MP_OKAY*)

Figure 8.4: Algorithm mp_add_d

Algorithm mp_add_d. This algorithm initiates a temporary mp_int with the value of the single digit and uses algorithm mp_add to add the two values together.

File: bn_mp_add_d.c

```

016
017  /* single digit addition */
018  int
019  mp_add_d (mp_int * a, mp_digit b, mp_int * c)
020  {
021      int      res, ix, oldused;
022      mp_digit *tmpa, *tmpc, mu;
023
024      /* grow c as required */
025      if (c->alloc < (a->used + 1)) {
026          if ((res = mp_grow(c, a->used + 1)) != MP_OKAY) {
027              return res;
028          }
029      }
030
031      /* if a is negative and |a| >= b, call c = |a| - b */
032      if ((a->sign == MP_NEG) && ((a->used > 1) || (a->dp[0] >= b))) {
033          /* temporarily fix sign of a */
034          a->sign = MP_ZPOS;
035
036          /* c = |a| - b */
037          res = mp_sub_d(a, b, c);
038
039          /* fix sign */
040          a->sign = c->sign = MP_NEG;
041
042          /* clamp */

```

```

043     mp_clamp(c);
044
045     return res;
046 }
047
048 /* old number of used digits in c */
049 oldused = c->used;
050
051 /* source alias */
052 tmpa    = a->dp;
053
054 /* destination alias */
055 tmpc    = c->dp;
056
057 /* if a is positive */
058 if (a->sign == MP_ZPOS) {
059     /* add digit, after this we're propagating
060      * the carry.
061      */
062     *tmpc  = *tmpa++ + b;
063     mu     = *tmpc >> DIGIT_BIT;
064     *tmpc++ &= MP_MASK;
065
066     /* now handle rest of the digits */
067     for (ix = 1; ix < a->used; ix++) {
068         *tmpc  = *tmpa++ + mu;
069         mu     = *tmpc >> DIGIT_BIT;
070         *tmpc++ &= MP_MASK;
071     }
072     /* set final carry */
073     ix++;
074     *tmpc++ = mu;
075
076     /* setup size */
077     c->used = a->used + 1;
078 } else {
079     /* a was negative and |a| < b */
080     c->used = 1;
081
082     /* the result is a single digit */
083     if (a->used == 1) {
084         *tmpc++ = b - a->dp[0];

```

```

085         } else {
086             *tmpc++ = b;
087         }
088
089         /* setup count so the clearing of oldused
090          * can fall through correctly
091          */
092         ix      = 1;
093     }
094
095     /* sign always positive */
096     c->sign = MP_ZPOS;
097
098     /* now zero to oldused */
099     while (ix++ < oldused) {
100         *tmpc++ = 0;
101     }
102     mp_clamp(c);
103
104     return MP_OKAY;
105 }
106
107 #endif
108
109 /* ref:          tag: v1.0.1 */
110 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
111 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Clever use of the letter 't'.

Subtraction

The single digit subtraction algorithm `mp_sub_d` is essentially the same except it uses `mp_sub` to subtract the digit from the `mp_int`.

8.2.2 Single Digit Multiplication

Single digit multiplication arises enough in division and radix conversion that it ought to be implement as a special case of the baseline multiplication algorithm. Essentially this algorithm is a modified version of algorithm `s_mp_mul_digs` where one of the multiplicands only has one digit.

Algorithm **mp_mul_d**.
Input. mp_int a and a mp_digit b
Output. $c = ab$

1. $pa \leftarrow a.used$
2. Grow c to at least $pa + 1$ digits.
3. $oldused \leftarrow c.used$
4. $c.used \leftarrow pa + 1$
5. $c.sign \leftarrow a.sign$
6. $\mu \leftarrow 0$
7. for ix from 0 to $pa - 1$ do
 - 7.1 $\hat{r} \leftarrow \mu + a_{ix}b$
 - 7.2 $c_{ix} \leftarrow \hat{r} \pmod{\beta}$
 - 7.3 $\mu \leftarrow \lfloor \hat{r}/\beta \rfloor$
8. $c_{pa} \leftarrow \mu$
9. for ix from $pa + 1$ to $oldused$ do
 - 9.1 $c_{ix} \leftarrow 0$
10. Clamp excess digits of c .
11. Return(MP_OKAY).

Figure 8.5: Algorithm mp_mul_d

Algorithm mp_mul_d. This algorithm quickly multiplies an mp_int by a small single digit value. It is specially tailored to the job and has a minimal of overhead. Unlike the full multiplication algorithms this algorithm does not require any significant temporary storage or memory allocations.

File: bn_mp_mul_d.c

```

016
017  /* multiply by a digit */
018  int
019  mp_mul_d (mp_int * a, mp_digit b, mp_int * c)
020  {
021      mp_digit u, *tmpa, *tmpc;
022      mp_word  r;
023      int      ix, res, olduse;
024
025      /* make sure c is big enough to hold a*b */
026      if (c->alloc < (a->used + 1)) {
027          if ((res = mp_grow (c, a->used + 1)) != MP_OKAY) {
028              return res;

```

```

029     }
030 }
031
032 /* get the original destinations used count */
033 olduse = c->used;
034
035 /* set the sign */
036 c->sign = a->sign;
037
038 /* alias for a->dp [source] */
039 tmpa = a->dp;
040
041 /* alias for c->dp [dest] */
042 tmpc = c->dp;
043
044 /* zero carry */
045 u = 0;
046
047 /* compute columns */
048 for (ix = 0; ix < a->used; ix++) {
049     /* compute product and carry sum for this term */
050     r      = (mp_word)u + ((mp_word)*tmpa++ * (mp_word)b);
051
052     /* mask off higher bits to get a single digit */
053     *tmpc++ = (mp_digit) (r & ((mp_word) MP_MASK));
054
055     /* send carry into next iteration */
056     u      = (mp_digit) (r >> ((mp_word) DIGIT_BIT));
057 }
058
059 /* store final carry [if any] and increment ix offset */
060 *tmpc++ = u;
061 ++ix;
062
063 /* now zero digits above the top */
064 while (ix++ < olduse) {
065     *tmpc++ = 0;
066 }
067
068 /* set used count */
069 c->used = a->used + 1;
070 mp_clamp(c);

```

```
071
072     return MP_OKAY;
073 }
074 #endif
075
076 /* ref:          tag: v1.0.1 */
077 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
078 /* commit time:  2017-08-29 22:27:36 +0200 */
```

In this implementation the destination c may point to the same `mp_int` as the source a since the result is written after the digit is read from the source. This function uses pointer aliases *tmpa* and *tmpc* for the digits of a and c respectively.

8.2.3 Single Digit Division

Like the single digit multiplication algorithm, single digit division is also a fairly common algorithm used in radix conversion. Since the divisor is only a single digit a specialized variant of the division algorithm can be used to compute the quotient.

Algorithm **mp_div_d**.

Input. mp_int a and a mp_digit b

Output. $c = \lfloor a/b \rfloor, d = a - cb$

1. If $b = 0$ then return(*MP_VAL*).
 2. If $b = 3$ then use algorithm mp_div_3 instead.
 3. Init q to $a.used$ digits.
 4. $q.used \leftarrow a.used$
 5. $q.sign \leftarrow a.sign$
 6. $\hat{w} \leftarrow 0$
 7. for ix from $a.used - 1$ down to 0 do
 - 7.1 $\hat{w} \leftarrow \hat{w}\beta + a_{ix}$
 - 7.2 If $\hat{w} \geq b$ then
 - 7.2.1 $t \leftarrow \lfloor \hat{w}/b \rfloor$
 - 7.2.2 $\hat{w} \leftarrow \hat{w} \pmod{b}$
 - 7.3 else
 - 7.3.1 $t \leftarrow 0$
 - 7.4 $q_{ix} \leftarrow t$
 8. $d \leftarrow \hat{w}$
 9. Clamp excess digits of q .
 10. $c \leftarrow q$
 11. Return(*MP_OKAY*).
-

Figure 8.6: Algorithm mp_div_d

Algorithm mp_div_d. This algorithm divides the mp_int a by the single mp_digit b using an optimized approach. Essentially in every iteration of the algorithm another digit of the dividend is reduced and another digit of quotient produced. Provided $b < \beta$ the value of \hat{w} after step 7.1 will be limited such that $0 \leq \lfloor \hat{w}/b \rfloor < \beta$.

If the divisor b is equal to three a variant of this algorithm is used which is called mp_div_3. It replaces the division by three with a multiplication by $\lfloor \beta/3 \rfloor$ and the appropriate shift and residual fixup. In essence it is much like the Barrett reduction from chapter seven.

File: bn_mp_div_d.c

```

016
017 static int s_is_power_of_two(mp_digit b, int *p)
018 {
019     int x;
020
```



```

021     /* fast return if no power of two */
022     if ((b == 0) || ((b & (b-1)) != 0)) {
023         return 0;
024     }
025
026     for (x = 0; x < DIGIT_BIT; x++) {
027         if (b == (((mp_digit)1)<<x)) {
028             *p = x;
029             return 1;
030         }
031     }
032     return 0;
033 }
034
035 /* single digit division (based on routine from MPI) */
036 int mp_div_d (mp_int * a, mp_digit b, mp_int * c, mp_digit * d)
037 {
038     mp_int q;
039     mp_word w;
040     mp_digit t;
041     int res, ix;
042
043     /* cannot divide by zero */
044     if (b == 0) {
045         return MP_VAL;
046     }
047
048     /* quick outs */
049     if ((b == 1) || (mp_iszero(a) == MP_YES)) {
050         if (d != NULL) {
051             *d = 0;
052         }
053         if (c != NULL) {
054             return mp_copy(a, c);
055         }
056         return MP_OKAY;
057     }
058
059     /* power of two ? */
060     if (s_is_power_of_two(b, &ix) == 1) {
061         if (d != NULL) {
062             *d = a->dp[0] & (((mp_digit)1)<<ix) - 1);

```

```

063     }
064     if (c != NULL) {
065         return mp_div_2d(a, ix, c, NULL);
066     }
067     return MP_OKAY;
068 }
069
070 #ifdef BN_MP_DIV_3_C
071     /* three? */
072     if (b == 3) {
073         return mp_div_3(a, c, d);
074     }
075 #endif
076
077     /* no easy answer [c'est la vie]. Just division */
078     if ((res = mp_init_size(&q, a->used)) != MP_OKAY) {
079         return res;
080     }
081
082     q.used = a->used;
083     q.sign = a->sign;
084     w = 0;
085     for (ix = a->used - 1; ix >= 0; ix--) {
086         w = (w << ((mp_word)DIGIT_BIT)) | ((mp_word)a->dp[ix]);
087
088         if (w >= b) {
089             t = (mp_digit)(w / b);
090             w -= ((mp_word)t) * ((mp_word)b);
091         } else {
092             t = 0;
093         }
094         q.dp[ix] = (mp_digit)t;
095     }
096
097     if (d != NULL) {
098         *d = (mp_digit)w;
099     }
100
101     if (c != NULL) {
102         mp_clamp(&q);
103         mp_exch(&q, c);
104     }

```

```

105     mp_clear(&q);
106
107     return res;
108 }
109
110 #endif
111
112 /* ref:          tag: v1.0.1 */
113 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
114 /* commit time: 2017-08-29 22:27:36 +0200 */

```

Like the implementation of algorithm `mp_div` this algorithm allows either of the quotient or remainder to be passed as a **NULL** pointer to indicate the respective value is not required. This allows a trivial single digit modular reduction algorithm, `mp_mod_d` to be created.

The division and remainder on lines 89 and 90 can be replaced often by a single division on most processors. For example, the 32-bit x86 based processors can divide a 64-bit quantity by a 32-bit quantity and produce the quotient and remainder simultaneously. Unfortunately the GCC compiler does not recognize that optimization and will actually produce two function calls to find the quotient and remainder respectively.

8.2.4 Single Digit Root Extraction

Finding the n 'th root of an integer is fairly easy as far as numerical analysis is concerned. Algorithms such as the Newton-Raphson approximation (8.6) series will converge very quickly to a root for any continuous function $f(x)$.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (8.6)$$

In this case the n 'th root is desired and $f(x) = x^n - a$ where a is the integer of which the root is desired. The derivative of $f(x)$ is simply $f'(x) = nx^{n-1}$. Of particular importance is that this algorithm will be used over the integers not over the a more continuous domain such as the real numbers. As a result the root found can be above the true root by few and must be manually adjusted. Ideally at the end of the algorithm the n 'th root b of an integer a is desired such that $b^n \leq a$.

Algorithm **mp_n.root**.
Input. mp_int a and a mp_digit b
Output. $c^b \leq a$

1. If b is even and $a.sign = MP_NEG$ return(MP_VAL).
2. $sign \leftarrow a.sign$
3. $a.sign \leftarrow MP_ZPOS$
4. $t2 \leftarrow 2$
5. Loop
 - 5.1 $t1 \leftarrow t2$
 - 5.2 $t3 \leftarrow t1^{b-1}$
 - 5.3 $t2 \leftarrow t3 \cdot t1$
 - 5.4 $t2 \leftarrow t2 - a$
 - 5.5 $t3 \leftarrow t3 \cdot b$
 - 5.6 $t3 \leftarrow \lfloor t2/t3 \rfloor$
 - 5.7 $t2 \leftarrow t1 - t3$
 - 5.8 If $t1 \neq t2$ then goto step 5.
6. Loop
 - 6.1 $t2 \leftarrow t1^b$
 - 6.2 If $t2 > a$ then
 - 6.2.1 $t1 \leftarrow t1 - 1$
 - 6.2.2 Goto step 6.
7. $a.sign \leftarrow sign$
8. $c \leftarrow t1$
9. $c.sign \leftarrow sign$
10. Return(MP_OKAY).

Figure 8.7: Algorithm mp_n.root

Algorithm mp_n.root. This algorithm finds the integer n 'th root of an input using the Newton-Raphson approach. It is partially optimized based on the observation that the numerator of $\frac{f(x)}{f'(x)}$ can be derived from a partial denominator. That is at first the denominator is calculated by finding x^{b-1} . This value can then be multiplied by x and have a subtracted from it to find the numerator. This saves a total of $b - 1$ multiplications by $t1$ inside the loop.

The initial value of the approximation is $t2 = 2$ which allows the algorithm to start with very small values and quickly converge on the root. Ideally this algorithm is meant to find the n 'th root of an input where n is bounded by $2 \leq n \leq 5$.

File: bn_mp_n_root.c

```
016
017  /* wrapper function for mp_n_root_ex()
018   * computes  $c = (a)^{1/b}$  such that  $(c)^b \leq a$  and  $(c+1)^b > a$ 
019   */
020  int mp_n_root (mp_int * a, mp_digit b, mp_int * c)
021  {
022      return mp_n_root_ex(a, b, c, 0);
023  }
024
025  #endif
026
027  /* ref:          tag: v1.0.1 */
028  /* git commit:  5953f62e42b24af93748b1ee5e1d062e242c2546 */
029  /* commit time: 2017-08-29 22:27:36 +0200 */
```

8.3 Random Number Generation

Random numbers come up in a variety of activities from public key cryptography to simple simulations and various randomized algorithms. Pollard-Rho factoring for example, can make use of random values as starting points to find factors of a composite integer. In this case the algorithm presented is solely for simulations and not intended for cryptographic use.

Algorithm **mp_rand**.

Input. An integer b

Output. A pseudo-random number of b digits

1. $a \leftarrow 0$
 2. If $b \leq 0$ return(*MP_OKAY*)
 3. Pick a non-zero random digit d .
 4. $a \leftarrow a + d$
 5. for ix from 1 to $d - 1$ do
 - 5.1 $a \leftarrow a \cdot \beta$
 - 5.2 Pick a random digit d .
 - 5.3 $a \leftarrow a + d$
 6. Return(*MP_OKAY*).
-

Figure 8.8: Algorithm mp_rand

Algorithm mp_rand. This algorithm produces a pseudo-random integer of b digits. By ensuring that the first digit is non-zero the algorithm also guarantees that the final result has at least b digits. It relies heavily on a third-part random number generator which should ideally generate uniformly all of the integers from 0 to $\beta - 1$.

File: bn_mp_rand.c

```

016
017  #if MP_GEN_RANDOM_MAX == 0xffffffff
018      #define MP_GEN_RANDOM_SHIFT  32
019  #elif MP_GEN_RANDOM_MAX == 32767
020      /* SHRT_MAX */
021      #define MP_GEN_RANDOM_SHIFT  15
022  #elif MP_GEN_RANDOM_MAX == 2147483647
023      /* INT_MAX */
024      #define MP_GEN_RANDOM_SHIFT  31
025  #elif !defined(MP_GEN_RANDOM_SHIFT)
026      #error Thou shalt define their own valid MP_GEN_RANDOM_SHIFT
027  #endif
028
029  /* makes a pseudo-random int of a given size */
030  static mp_digit s_gen_random(void)
031  {
032      mp_digit d = 0, msk = 0;
033      do {

```

```

034     d <= MP_GEN_RANDOM_SHIFT;
035     d |= ((mp_digit) MP_GEN_RANDOM());
036     msk <= MP_GEN_RANDOM_SHIFT;
037     msk |= (MP_MASK & MP_GEN_RANDOM_MAX);
038   } while ((MP_MASK & msk) != MP_MASK);
039   d &= MP_MASK;
040   return d;
041 }
042
043 int
044 mp_rand (mp_int * a, int digits)
045 {
046     int      res;
047     mp_digit d;
048
049     mp_zero (a);
050     if (digits <= 0) {
051         return MP_OKAY;
052     }
053
054     /* first place a random non-zero digit */
055     do {
056         d = s_gen_random();
057     } while (d == 0);
058
059     if ((res = mp_add_d (a, d, a)) != MP_OKAY) {
060         return res;
061     }
062
063     while (--digits > 0) {
064         if ((res = mp_lshd (a, 1)) != MP_OKAY) {
065             return res;
066         }
067
068         if ((res = mp_add_d (a, s_gen_random(), a)) != MP_OKAY) {
069             return res;
070         }
071     }
072
073     return MP_OKAY;
074 }
075 #endif

```

```
076
077  /* ref:          tag: v1.0.1 */
078  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
079  /* commit time: 2017-08-29 22:27:36 +0200 */
```

8.4 Formatted Representations

The ability to emit a radix- n textual representation of an integer is useful for interacting with human parties. For example, the ability to be given a string of characters such as “114585” and turn it into the radix- β equivalent would make it easier to enter numbers into a program.

8.4.1 Reading Radix- n Input

For the purposes of this text we will assume that a simple lower ASCII map (8.9) is used for the values of from 0 to 63 to printable characters. For example, when the character “N” is read it represents the integer 23. The first 16 characters of the map are for the common representations up to hexadecimal. After that they match the “base64” encoding scheme which are suitable chosen such that they are printable. While outputting as base64 may not be too helpful for human operators it does allow communication via non binary mediums.

| Value | Char | Value | Char | Value | Char | Value | Char |
|-------|------|-------|------|-------|------|-------|------|
| 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| 8 | 8 | 9 | 9 | 10 | A | 11 | B |
| 12 | C | 13 | D | 14 | E | 15 | F |
| 16 | G | 17 | H | 18 | I | 19 | J |
| 20 | K | 21 | L | 22 | M | 23 | N |
| 24 | O | 25 | P | 26 | Q | 27 | R |
| 28 | S | 29 | T | 30 | U | 31 | V |
| 32 | W | 33 | X | 34 | Y | 35 | Z |
| 36 | a | 37 | b | 38 | c | 39 | d |
| 40 | e | 41 | f | 42 | g | 43 | h |
| 44 | i | 45 | j | 46 | k | 47 | l |
| 48 | m | 49 | n | 50 | o | 51 | p |
| 52 | q | 53 | r | 54 | s | 55 | t |
| 56 | u | 57 | v | 58 | w | 59 | x |
| 60 | y | 61 | z | 62 | + | 63 | / |

Figure 8.9: Lower ASCII Map

Algorithm **mp_read_radix**.

Input. A string str of length sn and radix r .

Output. The radix- β equivalent mp_int .

1. If $r < 2$ or $r > 64$ return(MP_VAL).
 2. $ix \leftarrow 0$
 3. If $str_0 = \text{"-"}$ then do
 - 3.1 $ix \leftarrow ix + 1$
 - 3.2 $sign \leftarrow MP_NEG$
 4. else
 - 4.1 $sign \leftarrow MP_ZPOS$
 5. $a \leftarrow 0$
 6. for iy from ix to $sn - 1$ do
 - 6.1 Let y denote the position in the map of str_{iy} .
 - 6.2 If str_{iy} is not in the map or $y \geq r$ then goto step 7.
 - 6.3 $a \leftarrow a \cdot r$
 - 6.4 $a \leftarrow a + y$
 7. If $a \neq 0$ then $a.sign \leftarrow sign$
 8. Return(MP_OKAY).
-

Figure 8.10: Algorithm `mp_read_radix`

Algorithm `mp_read_radix`. This algorithm will read an ASCII string and produce the radix- β mp_int representation of the same integer. A minus symbol “-” may precede the string to indicate the value is negative, otherwise it is assumed to be positive. The algorithm will read up to sn characters from the input and will stop when it reads a character it cannot map the algorithm stops reading characters from the string. This allows numbers to be embedded as part of larger input without any significant problem.

File: `bn_mp_read_radix.c`

```

016
017  /* read a string [ASCII] in a given radix */
018  int mp_read_radix (mp_int * a, const char *str, int radix)
019  {
020      int      y, res, neg;
021      char     ch;
022
023      /* zero the digit bignum */
024      mp_zero(a);
025
```

```

026     /* make sure the radix is ok */
027     if ((radix < 2) || (radix > 64)) {
028         return MP_VAL;
029     }
030
031     /* if the leading digit is a
032      * minus set the sign to negative.
033      */
034     if (*str == '-') {
035         ++str;
036         neg = MP_NEG;
037     } else {
038         neg = MP_ZPOS;
039     }
040
041     /* set the integer to the default of zero */
042     mp_zero (a);
043
044     /* process each digit of the string */
045     while (*str != '\0') {
046         /* if the radix <= 36 the conversion is case insensitive
047          * this allows numbers like 1AB and 1ab to represent the same value
048          * [e.g. in hex]
049          */
050         ch = (radix <= 36) ? (char)toupper((int)*str) : *str;
051         for (y = 0; y < 64; y++) {
052             if (ch == mp_s_rmap[y]) {
053                 break;
054             }
055         }
056
057         /* if the char was found in the map
058          * and is less than the given radix add it
059          * to the number, otherwise exit the loop.
060          */
061         if (y < radix) {
062             if ((res = mp_mul_d (a, (mp_digit) radix, a)) != MP_OKAY) {
063                 return res;
064             }
065             if ((res = mp_add_d (a, (mp_digit) y, a)) != MP_OKAY) {
066                 return res;
067             }

```

```
068     } else {
069         break;
070     }
071     ++str;
072 }
073
074 /* set the sign only if a != 0 */
075 if (mp_iszero(a) != MP_YES) {
076     a->sign = neg;
077 }
078 return MP_OKAY;
079 }
080 #endif
081
082 /* ref:          tag: v1.0.1 */
083 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
084 /* commit time: 2017-08-29 22:27:36 +0200 */
```

8.4.2 Generating Radix- n Output

Generating radix- n output is fairly trivial with a division and remainder algorithm.

Algorithm **mp_toradix**.

Input. A mp_int a and an integer r

Output. The radix- r representation of a

1. If $r < 2$ or $r > 64$ return(MP_VAL).
 2. If $a = 0$ then $str = "0"$ and return(MP_OKAY).
 3. $t \leftarrow a$
 4. $str \leftarrow ""$
 5. if $t.sign = MP_NEG$ then
 - 5.1 $str \leftarrow str + "-"$
 - 5.2 $t.sign = MP_ZPOS$
 6. While ($t \neq 0$) do
 - 6.1 $d \leftarrow t \pmod{r}$
 - 6.2 $t \leftarrow \lfloor t/r \rfloor$
 - 6.3 Look up d in the map and store the equivalent character in y .
 - 6.4 $str \leftarrow str + y$
 7. If $str_0 = "-"$ then
 - 7.1 Reverse the digits $str_1, str_2, \dots, str_n$.
 8. Otherwise
 - 8.1 Reverse the digits $str_0, str_1, \dots, str_n$.
 9. Return(MP_OKAY).
-

Figure 8.11: Algorithm mp_toradix

Algorithm mp_toradix. This algorithm computes the radix- r representation of an mp_int a . The “digits” of the representation are extracted by reducing successive powers of $\lfloor a/r^k \rfloor$ the input modulo r until $r^k > a$. Note that instead of actually dividing by r^k in each iteration the quotient $\lfloor a/r \rfloor$ is saved for the next iteration. As a result a series of trivial $n \times 1$ divisions are required instead of a series of $n \times k$ divisions. One design flaw of this approach is that the digits are produced in the reverse order (see 8.12). To remedy this flaw the digits must be swapped or simply “reversed”.

File: bn_mp_toradix.c

```

016
017  /* stores a bignum as a ASCII string in a given radix (2..64) */
018  int mp_toradix (mp_int * a, char *str, int radix)
019  {
020      int      res, digs;
021      mp_int  t;
022      mp_digit d;

```

| Value of a | Value of d | Value of str |
|--------------|--------------|----------------|
| 1234 | — | — |
| 123 | 4 | “4” |
| 12 | 3 | “43” |
| 1 | 2 | “432” |
| 0 | 1 | “4321” |

Figure 8.12: Example of Algorithm `mp_toradix`.

```

023 char   *_s = str;
024
025 /* check range of the radix */
026 if ((radix < 2) || (radix > 64)) {
027     return MP_VAL;
028 }
029
030 /* quick out if its zero */
031 if (mp_iszero(a) == MP_YES) {
032     *str++ = '0';
033     *str = '\0';
034     return MP_OKAY;
035 }
036
037 if ((res = mp_init_copy (&t, a)) != MP_OKAY) {
038     return res;
039 }
040
041 /* if it is negative output a - */
042 if (t.sign == MP_NEG) {
043     ++_s;
044     *str++ = '-';
045     t.sign = MP_ZPOS;
046 }
047
048 digs = 0;
049 while (mp_iszero (&t) == MP_NO) {
050     if ((res = mp_div_d (&t, (mp_digit) radix, &t, &d)) != MP_OKAY) {
051         mp_clear (&t);
052         return res;
053     }

```

```
054     *str++ = mp_s_rmap[d];
055     ++digs;
056 }
057
058 /* reverse the digits of the string. In this case _s points
059  * to the first digit [exluding the sign] of the number]
060  */
061 bn_reverse ((unsigned char *)_s, digs);
062
063 /* append a NULL so the string is properly terminated */
064 *str = '\0';
065
066 mp_clear (&t);
067 return MP_OKAY;
068 }
069
070 #endif
071
072 /* ref:          tag: v1.0.1 */
073 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
074 /* commit time: 2017-08-29 22:27:36 +0200 */
```


Chapter 9

Number Theoretic Algorithms

This chapter discusses several fundamental number theoretic algorithms such as the greatest common divisor, least common multiple and Jacobi symbol computation. These algorithms arise as essential components in several key cryptographic algorithms such as the RSA public key algorithm and various Sieve based factoring algorithms.

9.1 Greatest Common Divisor

The greatest common divisor of two integers a and b , often denoted as (a, b) is the largest integer k that is a proper divisor of both a and b . That is, k is the largest integer such that $0 \equiv a \pmod{k}$ and $0 \equiv b \pmod{k}$ occur simultaneously.

The most common approach (cite) is to reduce one input modulo another. That is if a and b are divisible by some integer k and if $qa + r = b$ then r is also divisible by k . The reduction pattern follows $\langle a, b \rangle \rightarrow \langle b, a \bmod b \rangle$.



Figure 9.1: Algorithm Greatest Common Divisor (I)

This algorithm will quickly converge on the greatest common divisor since the residue r tends diminish rapidly. However, divisions are relatively expensive operations to perform and should ideally be avoided. There is another approach based on a similar relationship of greatest common divisors. The faster approach is based on the observation that if k divides both a and b it will also divide $a - b$. In particular, we would like $a - b$ to decrease in magnitude which implies that $b \geq a$.



Figure 9.2: Algorithm Greatest Common Divisor (II)

Proof *Algorithm 9.2 will return the greatest common divisor of a and b .* The algorithm in figure 9.2 will eventually terminate since $b \geq a$ the subtraction in step 1.2 will be a value less than b . In other words in every iteration that tuple $\langle a, b \rangle$ decrease in magnitude until eventually $a = b$. Since both a and b are always divisible by the greatest common divisor (*until the last iteration*) and in the last iteration of the algorithm $b = 0$, therefore, in the second to last iteration of the algorithm $b = a$ and clearly $(a, a) = a$ which concludes the proof. **QED.**

As a matter of practicality algorithm 9.1 decreases far too slowly to be useful.

Specially if b is much larger than a such that $b - a$ is still very much larger than a . A simple addition to the algorithm is to divide $b - a$ by a power of some integer p which does not divide the greatest common divisor but will divide $b - a$. In this case $\frac{b-a}{p}$ is also an integer and still divisible by the greatest common divisor.

However, instead of factoring $b - a$ to find a suitable value of p the powers of p can be removed from a and b that are in common first. Then inside the loop whenever $b - a$ is divisible by some power of p it can be safely removed.

Algorithm **Greatest Common Divisor (III)**.

Input. Two positive integers a and b greater than zero.

Output. The greatest common divisor (a, b) .

```

1.  $k \leftarrow 0$ 
2. While  $a$  and  $b$  are both divisible by  $p$  do
    2.1  $a \leftarrow \lfloor a/p \rfloor$ 
    2.2  $b \leftarrow \lfloor b/p \rfloor$ 
    2.3  $k \leftarrow k + 1$ 
3. While  $a$  is divisible by  $p$  do
    3.1  $a \leftarrow \lfloor a/p \rfloor$ 
4. While  $b$  is divisible by  $p$  do
    4.1  $b \leftarrow \lfloor b/p \rfloor$ 
5. While  $(b > 0)$  do
    5.1 Swap  $a$  and  $b$  such that  $a$  is the smallest of the two.
    5.2  $b \leftarrow b - a$ 
    5.3 While  $b$  is divisible by  $p$  do
        5.3.1  $b \leftarrow \lfloor b/p \rfloor$ 
6. Return( $a \cdot p^k$ ).
```

Figure 9.3: Algorithm Greatest Common Divisor (III)

This algorithm is based on the first except it removes powers of p first and inside the main loop to ensure the tuple $\langle a, b \rangle$ decreases more rapidly. The first loop on step two removes powers of p that are in common. A count, k , is kept which will present a common divisor of p^k . After step two the remaining common divisor of a and b cannot be divisible by p . This means that p can be safely divided out of the difference $b - a$ so long as the division leaves no remainder.

In particular the value of p should be chosen such that the division on step 5.3.1 occur often. It also helps that division by p be easy to compute. The ideal

choice of p is two since division by two amounts to a right logical shift. Another important observation is that by step five both a and b are odd. Therefore, the difference $b - a$ must be even which means that each iteration removes one bit from the largest of the pair.

9.1.1 Complete Greatest Common Divisor

The algorithms presented so far cannot handle inputs which are zero or negative. The following algorithm can handle all input cases properly and will produce the greatest common divisor.

Algorithm **mp_gcd**.

Input. mp_int a and b

Output. The greatest common divisor $c = (a, b)$.

1. If $a = 0$ then
 - 1.1 $c \leftarrow |b|$
 - 1.2 Return(*MP_OKAY*).
 2. If $b = 0$ then
 - 2.1 $c \leftarrow |a|$
 - 2.2 Return(*MP_OKAY*).
 3. $u \leftarrow |a|, v \leftarrow |b|$
 4. $k \leftarrow 0$
 5. While $u.used > 0$ and $v.used > 0$ and $u_0 \equiv v_0 \equiv 0 \pmod{2}$
 - 5.1 $k \leftarrow k + 1$
 - 5.2 $u \leftarrow \lfloor u/2 \rfloor$
 - 5.3 $v \leftarrow \lfloor v/2 \rfloor$
 6. While $u.used > 0$ and $u_0 \equiv 0 \pmod{2}$
 - 6.1 $u \leftarrow \lfloor u/2 \rfloor$
 7. While $v.used > 0$ and $v_0 \equiv 0 \pmod{2}$
 - 7.1 $v \leftarrow \lfloor v/2 \rfloor$
 8. While $v.used > 0$
 - 8.1 If $|u| > |v|$ then
 - 8.1.1 Swap u and v .
 - 8.2 $v \leftarrow |v| - |u|$
 - 8.3 While $v.used > 0$ and $v_0 \equiv 0 \pmod{2}$
 - 8.3.1 $v \leftarrow \lfloor v/2 \rfloor$
 9. $c \leftarrow u \cdot 2^k$
 10. Return(*MP_OKAY*).
-

Figure 9.4: Algorithm mp_gcd

Algorithm mp_gcd. This algorithm will produce the greatest common divisor of two mp_ints a and b . The algorithm was originally based on Algorithm B of Knuth [1, pp. 338] but has been modified to be simpler to explain. In theory it achieves the same asymptotic working time as Algorithm B and in practice this appears to be true.

The first two steps handle the cases where either one of or both inputs are zero. If either input is zero the greatest common divisor is the largest input or zero if they are both zero. If the inputs are not trivial then u and v are assigned the absolute values of a and b respectively and the algorithm will proceed to

reduce the pair.

Step five will divide out any common factors of two and keep track of the count in the variable k . After this step, two is no longer a factor of the remaining greatest common divisor between u and v and can be safely evenly divided out of either whenever they are even. Step six and seven ensure that the u and v respectively have no more factors of two. At most only one of the while-loops will iterate since they cannot both be even.

By step eight both of u and v are odd which is required for the inner logic. First the pair are swapped such that v is equal to or greater than u . This ensures that the subtraction on step 8.2 will always produce a positive and even result. Step 8.3 removes any factors of two from the difference u to ensure that in the next iteration of the loop both are once again odd.

After $v = 0$ occurs the variable u has the greatest common divisor of the pair $\langle u, v \rangle$ just after step six. The result must be adjusted by multiplying by the common factors of two (2^k) removed earlier.

File: bn_mp-gcd.c

```

016
017  /* Greatest Common Divisor using the binary method */
018  int mp_gcd (mp_int * a, mp_int * b, mp_int * c)
019  {
020      mp_int  u, v;
021      int     k, u_lsb, v_lsb, res;
022
023      /* either zero than gcd is the largest */
024      if (mp_iszero (a) == MP_YES) {
025          return mp_abs (b, c);
026      }
027      if (mp_iszero (b) == MP_YES) {
028          return mp_abs (a, c);
029      }
030
031      /* get copies of a and b we can modify */
032      if ((res = mp_init_copy (&u, a)) != MP_OKAY) {
033          return res;
034      }
035
036      if ((res = mp_init_copy (&v, b)) != MP_OKAY) {
037          goto LBL_U;
038      }
039

```

```

040     /* must be positive for the remainder of the algorithm */
041     u.sign = v.sign = MP_ZPOS;
042
043     /* B1. Find the common power of two for u and v */
044     u_lsb = mp_cnt_lsb(&u);
045     v_lsb = mp_cnt_lsb(&v);
046     k      = MIN(u_lsb, v_lsb);
047
048     if (k > 0) {
049         /* divide the power of two out */
050         if ((res = mp_div_2d(&u, k, &u, NULL)) != MP_OKAY) {
051             goto LBL_V;
052         }
053
054         if ((res = mp_div_2d(&v, k, &v, NULL)) != MP_OKAY) {
055             goto LBL_V;
056         }
057     }
058
059     /* divide any remaining factors of two out */
060     if (u_lsb != k) {
061         if ((res = mp_div_2d(&u, u_lsb - k, &u, NULL)) != MP_OKAY) {
062             goto LBL_V;
063         }
064     }
065
066     if (v_lsb != k) {
067         if ((res = mp_div_2d(&v, v_lsb - k, &v, NULL)) != MP_OKAY) {
068             goto LBL_V;
069         }
070     }
071
072     while (mp_iszero(&v) == MP_NO) {
073         /* make sure v is the largest */
074         if (mp_cmp_mag(&u, &v) == MP_GT) {
075             /* swap u and v to make sure v is >= u */
076             mp_exch(&u, &v);
077         }
078
079         /* subtract smallest from largest */
080         if ((res = s_mp_sub(&v, &u, &v)) != MP_OKAY) {
081             goto LBL_V;

```

```

082     }
083
084     /* Divide out all factors of two */
085     if ((res = mp_div_2d(&v, mp_cnt_lsb(&v), &v, NULL)) != MP_OKAY) {
086         goto LBL_V;
087     }
088 }
089
090 /* multiply by 2**k which we divided out at the beginning */
091 if ((res = mp_mul_2d (&u, k, c)) != MP_OKAY) {
092     goto LBL_V;
093 }
094 c->sign = MP_ZPOS;
095 res = MP_OKAY;
096 LBL_V:mp_clear (&u);
097 LBL_U:mp_clear (&v);
098 return res;
099 }
100 #endif
101
102 /* ref:          tag: v1.0.1 */
103 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
104 /* commit time: 2017-08-29 22:27:36 +0200 */

```

This function makes use of the macros `mp_iszero` and `mp_iseven`. The former evaluates to 1 if the input `mp_int` is equivalent to the integer zero otherwise it evaluates to 0. The latter evaluates to 1 if the input `mp_int` represents a non-zero even integer otherwise it evaluates to 0. Note that just because `mp_iseven` may evaluate to 0 does not mean the input is odd, it could also be zero. The three trivial cases of inputs are handled on lines 23 through 29. After those lines the inputs are assumed to be non-zero.

Lines 32 and 36 make local copies u and v of the inputs a and b respectively. At this point the common factors of two must be divided out of the two inputs. The block starting at line 43 removes common factors of two by first counting the number of trailing zero bits in both. The local integer k is used to keep track of how many factors of 2 are pulled out of both values. It is assumed that the number of factors will not exceed the maximum value of a C “int” data type¹.

At this point there are no more common factors of two in the two values. The divisions by a power of two on lines 61 and 67 remove any independent factors of

¹Strictly speaking no array in C may have more than entries than are accessible by an “int” so this is not a limitation.

two such that both u and v are guaranteed to be an odd integer before hitting the main body of the algorithm. The while loop on line 72 performs the reduction of the pair until v is equal to zero. The unsigned comparison and subtraction algorithms are used in place of the full signed routines since both values are guaranteed to be positive and the result of the subtraction is guaranteed to be non-negative.

9.2 Least Common Multiple

The least common multiple of a pair of integers is their product divided by their greatest common divisor. For two integers a and b the least common multiple is normally denoted as $[a, b]$ and numerically equivalent to $\frac{ab}{(a, b)}$. For example, if $a = 2 \cdot 2 \cdot 3 = 12$ and $b = 2 \cdot 3 \cdot 3 \cdot 7 = 126$ the least common multiple is $\frac{126}{(12, 126)} = \frac{126}{6} = 21$.

The least common multiple arises often in coding theory as well as number theory. If two functions have periods of a and b respectively they will collide, that is be in synchronous states, after only $[a, b]$ iterations. This is why, for example, random number generators based on Linear Feedback Shift Registers (LFSR) tend to use registers with periods which are co-prime (*e.g. the greatest common divisor is one.*). Similarly in number theory if a composite n has two prime factors p and q then maximal order of any unit of $\mathbb{Z}/n\mathbb{Z}$ will be $[p-1, q-1]$.

Algorithm **mp_lcm**.
Input. mp_int a and b
Output. The least common multiple $c = [a, b]$.

1. $c \leftarrow (a, b)$
2. $t \leftarrow a \cdot b$
3. $c \leftarrow \lfloor t/c \rfloor$
4. Return(*MP_OKAY*).

Figure 9.5: Algorithm mp_lcm

Algorithm mp_lcm. This algorithm computes the least common multiple of two mp_int inputs a and b . It computes the least common multiple directly by dividing the product of the two inputs by their greatest common divisor.

File: bn_mp_lcm.c
016

```

017  /* computes least common multiple as |a*b|/(a, b) */
018  int mp_lcm (mp_int * a, mp_int * b, mp_int * c)
019  {
020      int      res;
021      mp_int  t1, t2;
022
023
024      if ((res = mp_init_multi (&t1, &t2, NULL)) != MP_OKAY) {
025          return res;
026      }
027
028      /* t1 = get the GCD of the two inputs */
029      if ((res = mp_gcd (a, b, &t1)) != MP_OKAY) {
030          goto LBL_T;
031      }
032
033      /* divide the smallest by the GCD */
034      if (mp_cmp_mag(a, b) == MP_LT) {
035          /* store quotient in t2 such that t2 * b is the LCM */
036          if ((res = mp_div(a, &t1, &t2, NULL)) != MP_OKAY) {
037              goto LBL_T;
038          }
039          res = mp_mul(b, &t2, c);
040      } else {
041          /* store quotient in t2 such that t2 * a is the LCM */
042          if ((res = mp_div(b, &t1, &t2, NULL)) != MP_OKAY) {
043              goto LBL_T;
044          }
045          res = mp_mul(a, &t2, c);
046      }
047
048      /* fix the sign to positive */
049      c->sign = MP_ZPOS;
050
051  LBL_T:
052      mp_clear_multi (&t1, &t2, NULL);
053      return res;
054  }
055  #endif
056
057  /* ref:          tag: v1.0.1 */
058  /* git commit: 5953f62e42b24af93748b1ee5e1d062e242c2546 */

```

059 /* commit time: 2017-08-29 22:27:36 +0200 */

9.3 Jacobi Symbol Computation

To explain the Jacobi Symbol we shall first discuss the Legendre function² off which the Jacobi symbol is defined. The Legendre function computes whether or not an integer a is a quadratic residue modulo an odd prime p . Numerically it is equivalent to equation 9.1.

– *Tom, don't be an ass, cite your source here...!*

$$a^{(p-1)/2} \equiv \begin{cases} -1 & \text{if } a \text{ is a quadratic non-residue.} \\ 0 & \text{if } a \text{ divides } p. \\ 1 & \text{if } a \text{ is a quadratic residue.} \end{cases} \pmod{p} \quad (9.1)$$

Proof. Equation 9.1 correctly identifies the residue status of an integer a modulo a prime p . An integer a is a quadratic residue if the following equation has a solution.

$$x^2 \equiv a \pmod{p} \quad (9.2)$$

Consider the following equation.

$$0 \equiv x^{p-1} - 1 \equiv \left\{ (x^2)^{(p-1)/2} - a^{(p-1)/2} \right\} + \left(a^{(p-1)/2} - 1 \right) \pmod{p} \quad (9.3)$$

Whether equation 9.2 has a solution or not equation 9.3 is always true. If $a^{(p-1)/2} - 1 \equiv 0 \pmod{p}$ then the quantity in the braces must be zero. By reduction,

$$\begin{aligned} (x^2)^{(p-1)/2} - a^{(p-1)/2} &\equiv 0 \\ (x^2)^{(p-1)/2} &\equiv a^{(p-1)/2} \\ x^2 &\equiv a \pmod{p} \end{aligned} \quad (9.4)$$

As a result there must be a solution to the quadratic equation and in turn a must be a quadratic residue. If a does not divide p and a is not a quadratic residue then the only other value $a^{(p-1)/2}$ may be congruent to is -1 since

$$0 \equiv a^{p-1} - 1 \equiv (a^{(p-1)/2} + 1)(a^{(p-1)/2} - 1) \pmod{p} \quad (9.5)$$

One of the terms on the right hand side must be zero. **QED**

²Arrg. What is the name of this?

9.3.1 Jacobi Symbol

The Jacobi symbol is a generalization of the Legendre function for any odd non prime moduli p greater than 2. If $p = \prod_{i=0}^n p_i$ then the Jacobi symbol $\left(\frac{a}{p}\right)$ is equal to the following equation.

$$\left(\frac{a}{p}\right) = \left(\frac{a}{p_0}\right) \left(\frac{a}{p_1}\right) \cdots \left(\frac{a}{p_n}\right) \quad (9.6)$$

By inspection if p is prime the Jacobi symbol is equivalent to the Legendre function. The following facts³ will be used to derive an efficient Jacobi symbol algorithm. Where p is an odd integer greater than two and $a, b \in \mathbb{Z}$ the following are true.

1. $\left(\frac{a}{p}\right)$ equals -1 , 0 or 1 .
2. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$.
3. If $a \equiv b$ then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.
4. $\left(\frac{2}{p}\right)$ equals 1 if $p \equiv 1$ or $7 \pmod{8}$. Otherwise, it equals -1 .
5. $\left(\frac{a}{p}\right) \equiv \left(\frac{p}{a}\right) \cdot (-1)^{(p-1)(a-1)/4}$. More specifically $\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right)$ if $p \equiv a \equiv 1 \pmod{4}$.

Using these facts if $a = 2^k \cdot a'$ then

$$\begin{aligned} \left(\frac{a}{p}\right) &= \left(\frac{2^k}{p}\right) \left(\frac{a'}{p}\right) \\ &= \left(\frac{2}{p}\right)^k \left(\frac{a'}{p}\right) \end{aligned} \quad (9.7)$$

By fact five,

$$\left(\frac{a}{p}\right) = \left(\frac{p}{a}\right) \cdot (-1)^{(p-1)(a-1)/4} \quad (9.8)$$

³See HAC [2, pp. 72-74] for further details.

Subsequently by fact three since $p \equiv (p \bmod a) \pmod{a}$ then

$$\left(\frac{a}{p}\right) = \left(\frac{p \bmod a}{a}\right) \cdot (-1)^{(p-1)(a-1)/4} \quad (9.9)$$

By putting both observations into equation 9.7 the following simplified equation is formed.

$$\left(\frac{a}{p}\right) = \left(\frac{2}{p}\right)^k \left(\frac{p \bmod a'}{a'}\right) \cdot (-1)^{(p-1)(a'-1)/4} \quad (9.10)$$

The value of $\left(\frac{p \bmod a'}{a'}\right)$ can be found by using the same equation recursively. The value of $\left(\frac{2}{p}\right)^k$ equals 1 if k is even otherwise it equals $\left(\frac{2}{p}\right)$. Using this approach the factors of p do not have to be known. Furthermore, if $(a, p) = 1$ then the algorithm will terminate when the recursion requests the Jacobi symbol computation of $\left(\frac{1}{a'}\right)$ which is simply 1.

Algorithm **mp_jacobi**.

Input. mp_int a and p , $a \geq 0$, $p \geq 3$, $p \equiv 1 \pmod{2}$

Output. The Jacobi symbol $c = \left(\frac{a}{p}\right)$.

1. If $a = 0$ then
 - 1.1 $c \leftarrow 0$
 - 1.2 Return(*MP_OKAY*).
 2. If $a = 1$ then
 - 2.1 $c \leftarrow 1$
 - 2.2 Return(*MP_OKAY*).
 3. $a' \leftarrow a$
 4. $k \leftarrow 0$
 5. While $a'.used > 0$ and $a'_0 \equiv 0 \pmod{2}$
 - 5.1 $k \leftarrow k + 1$
 - 5.2 $a' \leftarrow \lfloor a'/2 \rfloor$
 6. If $k \equiv 0 \pmod{2}$ then
 - 6.1 $s \leftarrow 1$
 7. else
 - 7.1 $r \leftarrow p_0 \pmod{8}$
 - 7.2 If $r = 1$ or $r = 7$ then
 - 7.2.1 $s \leftarrow 1$
 - 7.3 else
 - 7.3.1 $s \leftarrow -1$
 8. If $p_0 \equiv a'_0 \equiv 3 \pmod{4}$ then
 - 8.1 $s \leftarrow -s$
 9. If $a' \neq 1$ then
 - 9.1 $p' \leftarrow p \pmod{a'}$
 - 9.2 $s \leftarrow s \cdot \text{mp_jacobi}(p', a')$
 10. $c \leftarrow s$
 11. Return(*MP_OKAY*).
-

Figure 9.6: Algorithm mp_jacobi

Algorithm mp_jacobi. This algorithm computes the Jacobi symbol for an arbitrary positive integer a with respect to an odd integer p greater than three. The algorithm is based on algorithm 2.149 of HAC [2, pp. 73].

Step numbers one and two handle the trivial cases of $a = 0$ and $a = 1$ respectively. Step five determines the number of two factors in the input a . If k is even then the term $\left(\frac{2}{p}\right)^k$ must always evaluate to one. If k is odd then the term evaluates to one if p_0 is congruent to one or seven modulo eight, otherwise

it evaluates to -1 . After the the $\left(\frac{2}{p}\right)^k$ term is handled the $(-1)^{(p-1)(a'-1)/4}$ is computed and multiplied against the current product s . The latter term evaluates to one if both p and a' are congruent to one modulo four, otherwise it evaluates to negative one.

By step nine if a' does not equal one a recursion is required. Step 9.1 computes $p' \equiv p \pmod{a'}$ and will recurse to compute $\left(\frac{p'}{a'}\right)$ which is multiplied against the current Jacobi product.

File: bn_mp_jacobi.c

```

016
017  /* computes the jacobi c = (a | n) (or Legendre if n is prime)
018   * HAC pp. 73 Algorithm 2.149
019   * HAC is wrong here, as the special case of (0 | 1) is not
020   * handled correctly.
021   */
022  int mp_jacobi (mp_int * a, mp_int * n, int *c)
023  {
024      mp_int  a1, p1;
025      int      k, s, r, res;
026      mp_digit residue;
027
028      /* if a < 0 return MP_VAL */
029      if (mp_isneg(a) == MP_YES) {
030          return MP_VAL;
031      }
032
033      /* if n <= 0 return MP_VAL */
034      if (mp_cmp_d(n, 0) != MP_GT) {
035          return MP_VAL;
036      }
037
038      /* step 1. handle case of a == 0 */
039      if (mp_iszero(a) == MP_YES) {
040          /* special case of a == 0 and n == 1 */
041          if (mp_cmp_d(n, 1) == MP_EQ) {
042              *c = 1;
043          } else {
044              *c = 0;
045          }
046          return MP_OKAY;
047      }

```

```

048
049 /* step 2.  if a == 1, return 1 */
050 if (mp_cmp_d (a, 1) == MP_EQ) {
051     *c = 1;
052     return MP_OKAY;
053 }
054
055 /* default */
056 s = 0;
057
058 /* step 3.  write a = a1 * 2**k */
059 if ((res = mp_init_copy (&a1, a)) != MP_OKAY) {
060     return res;
061 }
062
063 if ((res = mp_init (&p1)) != MP_OKAY) {
064     goto LBL_A1;
065 }
066
067 /* divide out larger power of two */
068 k = mp_cnt_lsb(&a1);
069 if ((res = mp_div_2d(&a1, k, &a1, NULL)) != MP_OKAY) {
070     goto LBL_P1;
071 }
072
073 /* step 4.  if e is even set s=1 */
074 if ((k & 1) == 0) {
075     s = 1;
076 } else {
077     /* else set s=1 if p = 1/7 (mod 8) or s=-1 if p = 3/5 (mod 8) */
078     residue = n->dp[0] & 7;
079
080     if ((residue == 1) || (residue == 7)) {
081         s = 1;
082     } else if ((residue == 3) || (residue == 5)) {
083         s = -1;
084     }
085 }
086
087 /* step 5.  if p == 3 (mod 4) *and* a1 == 3 (mod 4) then s = -s */
088 if ( ((n->dp[0] & 3) == 3) && ((a1.dp[0] & 3) == 3) ) {
089     s = -s;

```



```

090     }
091
092     /* if a1 == 1 we're done */
093     if (mp_cmp_d (&a1, 1) == MP_EQ) {
094         *c = s;
095     } else {
096         /* n1 = n mod a1 */
097         if ((res = mp_mod (n, &a1, &p1)) != MP_OKAY) {
098             goto LBL_P1;
099         }
100         if ((res = mp_jacobi (&p1, &a1, &r)) != MP_OKAY) {
101             goto LBL_P1;
102         }
103         *c = s * r;
104     }
105
106     /* done */
107     res = MP_OKAY;
108 LBL_P1:mp_clear (&p1);
109 LBL_A1:mp_clear (&a1);
110     return res;
111 }
112 #endif
113
114 /* ref:          tag: v1.0.1 */
115 /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
116 /* commit time: 2017-08-29 22:27:36 +0200 */

```

As a matter of practicality the variable a' as per the pseudo-code is represented by the variable $a1$ since the $'$ symbol is not valid for a C variable name character.

The two simple cases of $a = 0$ and $a = 1$ are handled at the very beginning to simplify the algorithm. If the input is non-trivial the algorithm has to proceed compute the Jacobi. The variable s is used to hold the current Jacobi product. Note that s is merely a C “int” data type since the values it may obtain are merely -1 , 0 and 1 .

After a local copy of a is made all of the factors of two are divided out and the total stored in k . Technically only the least significant bit of k is required, however, it makes the algorithm simpler to follow to perform an addition. In practice an exclusive-or and addition have the same processor requirements and neither is faster than the other.

Line 59 through 71 determines the value of $\left(\frac{2}{p}\right)^k$. If the least significant bit

of k is zero than k is even and the value is one. Otherwise, the value of s depends on which residue class p belongs to modulo eight. The value of $(-1)^{(p-1)(a'-1)/4}$ is compute and multiplied against s on lines 73 through 76.

Finally, if $a1$ does not equal one the algorithm must recurse and compute $\left(\frac{p'}{a'}\right)$.

– *Comment about default s and such...*

9.4 Modular Inverse

The modular inverse of a number actually refers to the modular multiplicative inverse. Essentially for any integer a such that $(a, p) = 1$ there exist another integer b such that $ab \equiv 1 \pmod{p}$. The integer b is called the multiplicative inverse of a which is denoted as $b = a^{-1}$. Technically speaking modular inversion is a well defined operation for any finite ring or field not just for rings and fields of integers. However, the former will be the matter of discussion.

The simplest approach is to compute the algebraic inverse of the input. That is to compute $b \equiv a^{\Phi(p)-1}$. If $\Phi(p)$ is the order of the multiplicative subgroup modulo p then b must be the multiplicative inverse of a . The proof of which is trivial.

$$ab \equiv a \left(a^{\Phi(p)-1} \right) \equiv a^{\Phi(p)} \equiv a^0 \equiv 1 \pmod{p} \quad (9.11)$$

However, as simple as this approach may be it has two serious flaws. It requires that the value of $\Phi(p)$ be known which if p is composite requires all of the prime factors. This approach also is very slow as the size of p grows.

A simpler approach is based on the observation that solving for the multiplicative inverse is equivalent to solving the linear Diophantine⁴ equation.

$$ab + pq = 1 \quad (9.12)$$

Where a , b , p and q are all integers. If such a pair of integers $\langle b, q \rangle$ exist than b is the multiplicative inverse of a modulo p . The extended Euclidean algorithm (Knuth [1, pp. 342]) can be used to solve such equations provided $(a, p) = 1$. However, instead of using that algorithm directly a variant known as the binary Extended Euclidean algorithm will be used in its place. The binary approach is very similar to the binary greatest common divisor algorithm except it will produce a full solution to the Diophantine equation.

⁴See LeVeque [20, pp. 40-43] for more information.

9.4.1 General Case

Algorithm **mp_invmod**.

Input. mp_int a and b , $(a, b) = 1$, $p \geq 2$, $0 < a < p$.

Output. The modular inverse $c \equiv a^{-1} \pmod{b}$.

1. If $b \leq 0$ then return(*MP_VAL*).
 2. If $b_0 \equiv 1 \pmod{2}$ then use algorithm *fast_mp_invmod*.
 3. $x \leftarrow |a|, y \leftarrow b$
 4. If $x_0 \equiv y_0 \equiv 0 \pmod{2}$ then return(*MP_VAL*).
 5. $B \leftarrow 0, C \leftarrow 0, A \leftarrow 1, D \leftarrow 1$
 6. While $u.used > 0$ and $u_0 \equiv 0 \pmod{2}$
 - 6.1 $u \leftarrow \lfloor u/2 \rfloor$
 - 6.2 If $(A.used > 0$ and $A_0 \equiv 1 \pmod{2})$ or $(B.used > 0$ and $B_0 \equiv 1 \pmod{2})$ then
 - 6.2.1 $A \leftarrow A + y$
 - 6.2.2 $B \leftarrow B - x$
 - 6.3 $A \leftarrow \lfloor A/2 \rfloor$
 - 6.4 $B \leftarrow \lfloor B/2 \rfloor$
 7. While $v.used > 0$ and $v_0 \equiv 0 \pmod{2}$
 - 7.1 $v \leftarrow \lfloor v/2 \rfloor$
 - 7.2 If $(C.used > 0$ and $C_0 \equiv 1 \pmod{2})$ or $(D.used > 0$ and $D_0 \equiv 1 \pmod{2})$ then
 - 7.2.1 $C \leftarrow C + y$
 - 7.2.2 $D \leftarrow D - x$
 - 7.3 $C \leftarrow \lfloor C/2 \rfloor$
 - 7.4 $D \leftarrow \lfloor D/2 \rfloor$
 8. If $u \geq v$ then
 - 8.1 $u \leftarrow u - v$
 - 8.2 $A \leftarrow A - C$
 - 8.3 $B \leftarrow B - D$
 9. else
 - 9.1 $v \leftarrow v - u$
 - 9.2 $C \leftarrow C - A$
 - 9.3 $D \leftarrow D - B$
 10. If $u \neq 0$ goto step 6.
 11. If $v \neq 1$ return(*MP_VAL*).
 12. While $C \leq 0$ do
 - 12.1 $C \leftarrow C + b$
 13. While $C \geq b$ do
 - 13.1 $C \leftarrow C - b$
 14. $c \leftarrow C$
 15. Return(*MP_OKAY*).
-

Algorithm mp_invmod. This algorithm computes the modular multi-

plicative inverse of an integer a modulo an integer b . This algorithm is a variation of the extended binary Euclidean algorithm from HAC [2, pp. 608]. It has been modified to only compute the modular inverse and not a complete Diophantine solution.

If $b \leq 0$ then the modulus is invalid and MP_VAL is returned. Similarly if both a and b are even then there cannot be a multiplicative inverse for a and the error is reported.

The astute reader will observe that steps seven through nine are very similar to the binary greatest common divisor algorithm mp_gcd. In this case the other variables to the Diophantine equation are solved. The algorithm terminates when $u = 0$ in which case the solution is

$$Ca + Db = v \quad (9.13)$$

If v , the greatest common divisor of a and b is not equal to one then the algorithm will report an error as no inverse exists. Otherwise, C is the modular inverse of a . The actual value of C is congruent to, but not necessarily equal to, the ideal modular inverse which should lie within $1 \leq a^{-1} < b$. Step numbers twelve and thirteen adjust the inverse until it is in range. If the original input a is within $0 < a < p$ then only a couple of additions or subtractions will be required to adjust the inverse.

File: bn_mp_invmod.c

```

016
017  /* hac 14.61, pp608 */
018  int mp_invmod (mp_int * a, mp_int * b, mp_int * c)
019  {
020      /* b cannot be negative */
021      if ((b->sign == MP_NEG) || (mp_iszero(b) == MP_YES)) {
022          return MP_VAL;
023      }
024
025      #ifndef BN_FAST_MP_INVMOD_C
026          /* if the modulus is odd we can use a faster routine instead */
027          if ((mp_isodd(b) == MP_YES) && (mp_cmp_d(b, 1) != MP_EQ)) {
028              return fast_mp_invmod (a, b, c);
029          }
030      #endif
031
032      #ifdef BN_MP_INVMOD_SLOW_C
033          return mp_invmod_slow(a, b, c);

```

```

034  #else
035      return MP_VAL;
036  #endif
037  }
038  #endif
039
040  /* ref:          tag: v1.0.1 */
041  /* git commit:   5953f62e42b24af93748b1ee5e1d062e242c2546 */
042  /* commit time:  2017-08-29 22:27:36 +0200 */

```

Odd Moduli

When the modulus b is odd the variables A and C are fixed and are not required to compute the inverse. In particular by attempting to solve the Diophantine $Cb + Da = 1$ only B and D are required to find the inverse of a .

The algorithm `fast_mp_invmod` is a direct adaptation of algorithm `mp_invmod` with all all steps involving either A or C removed. This optimization will halve the time required to compute the modular inverse.

9.5 Primality Tests

A non-zero integer a is said to be prime if it is not divisible by any other integer excluding one and itself. For example, $a = 7$ is prime since the integers $2 \dots 6$ do not evenly divide a . By contrast, $a = 6$ is not prime since $a = 6 = 2 \cdot 3$.

Prime numbers arise in cryptography considerably as they allow finite fields to be formed. The ability to determine whether an integer is prime or not quickly has been a viable subject in cryptography and number theory for considerable time. The algorithms that will be presented are all probabilistic algorithms in that when they report an integer is composite it must be composite. However, when the algorithms report an integer is prime the algorithm may be incorrect.

As will be discussed it is possible to limit the probability of error so well that for practical purposes the probability of error might as well be zero. For the purposes of these discussions let n represent the candidate integer of which the primality is in question.

9.5.1 Trial Division

Trial division means to attempt to evenly divide a candidate integer by small prime integers. If the candidate can be evenly divided it obviously cannot be

prime. By dividing by all primes $1 < p \leq \sqrt{n}$ this test can actually prove whether an integer is prime. However, such a test would require a prohibitive amount of time as n grows.

Instead of dividing by every prime, a smaller, more manageable set of primes may be used instead. By performing trial division with only a subset of the primes less than $\sqrt{n} + 1$ the algorithm cannot prove if a candidate is prime. However, often it can prove a candidate is not prime.

The benefit of this test is that trial division by small values is fairly efficient. Specially compared to the other algorithms that will be discussed shortly. The probability that this approach correctly identifies a composite candidate when tested with all primes upto q is given by $1 - \frac{1.12}{\ln(q)}$. The graph (9.7) demonstrates the probability of success for the range $3 \leq q \leq 100$.

At approximately $q = 30$ the gain of performing further tests diminishes fairly quickly. At $q = 90$ further testing is generally not going to be of any practical use. In the case of LibTomMath the default limit $q = 256$ was chosen since it is not too high and will eliminate approximately 80% of all candidate integers. The constant **PRIME_SIZE** is equal to the number of primes in the test base. The array `_prime_tab` is an array of the first **PRIME_SIZE** prime numbers.

Algorithm mp_prime_is_divisible. This algorithm attempts to determine if a candidate integer n is composite by performing trial divisions.

File: bn_mp_prime_is_divisible.c

```

016
017  /* determines if an integers is divisible by one
018   * of the first PRIME_SIZE primes or not
019   *
020   * sets result to 0 if not, 1 if yes
021   */
022  int mp_prime_is_divisible (mp_int * a, int *result)
023  {
024      int      err, ix;
025      mp_digit res;
026
027      /* default to not */
028      *result = MP_NO;
029
030      for (ix = 0; ix < PRIME_SIZE; ix++) {
```